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## ON BASICITY OF EIGENFUNCTIONS OF SECOND ORDER DISCONTINUOUS DIFFERENTIAL OPERATOR

B.T. BILALOV, T.B. GASYMOV

**Abstract.** We consider a spectral problem for a second order discontinuous differential operator with spectral parameter in the boundary condition. We present a method for establishing the basicity of eigenfunctions for such problem. We also consider a direct expansion of a Banach space with respect to subspaces and we propose a method for constructing a basis for a space by the bases in subspaces. We also consider the cases when the bases for subspaces are isomorphic and the corresponding isomorphisms are not needed. The completeness, minimality and uniform minimality of the corresponding systems are studied. This approach has extensive applications in the spectral theory of discontinuous differential operators.

**Keywords:** eigenfunctions, basis, completeness, minimality, uniform minimality.

## 1. INTRODUCTION

The study of the spectral properties of many discrete differential operators requires new methods for constructing bases. This was the motivations for many mathematicians to study intensively the basis properties (such as completeness, minimality, basicity) of the systems of special functions, mostly eigenfunctions and associated functions of differential operators. Various methods were developed for establishing these properties. For more information, we refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9].

In the case of a discontinuous differential operator, there arise the systems of eigenfunctions whose basicity can not be treated by the standard methods. To shed some light on this situation, we consider the following model spectral problem for a second order discontinuous differential operator

$$-y''(x) = \lambda y(x), \quad x \in (-1, 0) \cup (0, 1), \quad (1)$$

with the boundary conditions

$$y(-1) = y(1) = 0, \quad y(-0) = y(+0), \quad y'(-0) - y'(0) = \lambda m y(0). \quad (2)$$

This spectral problem has two sets of the eigenfunctions [10]:

$$u_{1n}(x) = \sin \pi n x, \quad x \in [-1, 1], \quad n \in \mathbb{N},$$

and

$$\tilde{u}_{2n}(x) = \begin{cases} \sin \pi n x + o\left(\frac{1}{n}\right), & x \in [-1, 0], \\ -\sin \pi n x + o\left(\frac{1}{n}\right), & x \in [0, 1], \quad n \in \mathbb{N}. \end{cases}$$

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Б.Т. БИЛАЛОВ, Т.Б. КАСУМОВ О БАЗИСНОСТИ СОБСТВЕННЫХ ФУНКЦИЙ РАЗРЫВНОГО ДИФФЕРЕНЦИАЛЬНОГО ОПЕРАТОРА ВТОРОГО ПОРЯДКА.

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Such spectral problems arise while one solves the problem of a loaded string fixed at both ends with a load placed in the middle of the string by the Fourier method [11, 12]. The use of this method requires the study of basis properties of the double system  $\{u_{1n}; \tilde{u}_{2n}\}_{n \in \mathbb{N}}$  in corresponding spaces of functions (usually in the Lebesgue or Sobolev spaces). Of course, it should be started with the basis properties of the system  $\{u_{1n}; u_{2n}\}_{n \in \mathbb{N}}$ , which is the principal part of the asymptotics of the system  $\{u_{1n}; \tilde{u}_{2n}\}_{n \in \mathbb{N}}$ :

$$u_{2n}(x) = \begin{cases} \sin \pi n x, & [-1, 0), \\ -\sin \pi n x, & x \in [0, 1]. \end{cases}$$

This is usually done by applying various perturbation methods. This approach is well studied (see, e.g., the articles [6, 7, 8, 9, 14, 15, 16, 17, 18, 19, 28, 29, 30, 31] and the monographs [13, 20, 21, 22, 23, 24]). On the other hand, it is not difficult to see that the principal part  $\{u_{1n}; \vartheta_n\}_{n \in \mathbb{N}}$  is not a standard (in other words, classical) system. It turns out that the form of the system  $\{u_{1n}; \vartheta_n\}_{n \in \mathbb{N}}$  is not special, i.e. it can be derived from the general case. A general approach applied to these systems allows one to introduce a new method for constructing bases with a lot of applications in the spectral theory of differential operators.

In this work, we consider an abstract approach to the above problem. We consider a direct expansion of a Banach space with respect to subspaces. We propose a method for constructing a basis for a space by means of the bases in the subspaces. We also consider the cases, when the bases in the subspaces are isomorphic and the corresponding isomorphisms may not hold. And we study the completeness, the minimality and the uniform minimality of the corresponding systems. The obtained results are applied to proving the basicity for the eigenfunctions of a second order discontinuous differential operator with the spectral parameter in the boundary condition.

## 2. NOTATION AND PRELIMINARIES

We will use the standard notation. Namely,  $\mathbb{N}$  is a set of all positive integers,  $L[M]$  denotes the linear span of the set  $M$  and  $\overline{M}$  stands for the closure of  $M$ ;  $X^*$  denotes the dual space for  $X$ . Let  $L(X_1, X_2)$  be a space of linear bounded operators from  $X_1$  into  $X_2$  with  $L(X, X) = L(X)$ ,  $D_T$  be the domain of an operator  $T$  and  $R_T$  be the range of  $T$ . Let  $\text{Ker } T$  stand for the kernel of an operator  $T$ ,  $\langle x, f \rangle = f(x)$  denote the value of a functional  $f$  at a point  $x$ . Banach space are referred to as  $B$ -space; Hilbert space is referred to as  $H$ -space;  $\|\cdot\|_X$  denotes a norm in  $X$ ;  $\Leftrightarrow$  means "if and only if";  $1:n \equiv \{1; \dots; n\}$ ;  $\delta_{ij}$  is the Kronecker delta.

Let us recall the definitions of completeness, minimality, uniform minimality, basicity and basicity with parentheses of a system in a  $B$ -space.

*Let  $X$  be some  $B$ -space. A system  $\{u_n\}_{n \in \mathbb{N}} \subset X$  is called complete in  $X$  if  $L[\overline{\{u_n\}_{n \in \mathbb{N}}}] = X$ .*

Using the Hahn-Banach theorem, it is easy to prove the validity of the following completeness criterion.

**Completeness criterion.** *Let  $X$  be a normed space. A system  $\{x_n\}_{n \in \mathbb{N}} \subset X$  is complete in  $X$  if and only if for all  $f \in X^* : \langle x_n, f \rangle = 0$  for each  $n \in \mathbb{N}$ , implies  $f = 0$ .*

*A system  $\{x_n\}_{n \in \mathbb{N}} \subset X$  is called minimal in  $X$  if*

$$x_k \notin \overline{L[\{x_n\}_{n \in \mathbb{N}_k}]} \quad \text{for all } k \in \mathbb{N}, \quad \text{where } \mathbb{N}_k = \mathbb{N} \setminus \{k\}.$$

*Systems  $\{x_n\}_{n \in \mathbb{N}} \subset X$  and  $\{x_n^*\}_{n \in \mathbb{N}} \subset X^*$  are called biorthogonal if  $\langle x_m, x_n^* \rangle = \delta_{nm}$  for all  $n, m \in \mathbb{N}$ .*

The following minimality criterion is available in many monographs.

**Minimality criterion.** *A system in a  $B$ -space is minimal if and only if it has a biorthogonal system.*

The following basicity criterion in  $B$ -spaces is true.

**Basicity criterion.** A system  $\{x_n\}_{n \in \mathbb{N}} \subset X$  forms a basis for a B-space  $X$  if and only if the following conditions are satisfied:

- 1)  $\{x_n\}_{n \in \mathbb{N}}$  is complete in  $X$ ;
- 2)  $\{x_n\}_{n \in \mathbb{N}}$  is minimal in  $X$ ;
- 3) The projectors  $P_m(\cdot) = \sum_{n=1}^m \langle \cdot, x_n^* \rangle x_n$  are uniformly bounded, i.e., there exists  $M > 0$  such that

$$\|P_m x\|_X \leq M \|x\|_X, \quad \forall x \in X,$$

where  $\{x_n^*\}_{n \in \mathbb{N}} \subset X^*$  is a system biorthogonal to  $\{x_n\}_{n \in \mathbb{N}}$ .

A system  $\{x_n\}_{n \in \mathbb{N}} \subset X$  is called uniformly minimal in  $X$  if

$$\exists \delta > 0 : \quad \inf_{\forall u \in L[\{x_n\}_{n \neq k}]} \|x_k - u\|_X \geq \delta \|x_k\|_X, \quad \forall k \in \mathbb{N}.$$

**Uniform minimality criterion.** A complete system  $\{x_n\}_{n \in \mathbb{N}} \subset X$  is uniformly minimal in  $X$  if and only if  $\sup_n \|x_n\|_X \|x_n^*\|_{X^*} < +\infty$ , where  $\{x_n^*\}_{n \in \mathbb{N}} \subset X^*$  is the biorthogonal system.

If a system  $\{x_n\}_{n \in \mathbb{N}} \subset X$  forms a basis in  $X$ , then it is uniformly minimal.

We will also need the concept of basicity with parentheses.

A system  $\{x_n\}_{n \in \mathbb{N}} \subset X$  is called a basis with parentheses for  $X$  if there exists a sequence of indices  $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N} : n_k < n_{k+1}, \forall k \in \mathbb{N}$ , such that every  $x \in X$  can be uniquely expanded in a series

$$x = \sum_{k=0}^{\infty} \sum_{j=n_k+1}^{n_{k+1}} c_j x_j, \quad n_0 = 0.$$

**Basicity with parentheses criterion.** A system  $\{x_n\}_{n \in \mathbb{N}}$  forms a basis with parentheses for a B-space  $X$  if and only if the following conditions are satisfied:

- 1)  $\{x_n\}_{n \in \mathbb{N}}$  is complete in  $X$ ;
- 2)  $\{x_n\}_{n \in \mathbb{N}}$  is minimal in  $X$ ;
- 3) there exists a sequence of indices  $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$  obeying  $n_k < n_{k+1}$  for all  $k \in \mathbb{N}$  such that the projectors  $Q_k(\cdot) = \sum_{n=1}^{n_k} \langle \cdot, x_n^* \rangle x_n$  are uniformly bounded, i.e., there exists  $M > 0$  such that

$$\|Q_k x\|_X \leq M \|x\|_X \quad \text{for all } x \in X,$$

where  $\{x_n^*\}_{n \in \mathbb{N}} \subset X^*$  is a system biorthogonal to  $\{x_n\}_{n \in \mathbb{N}}$ .

For more information we refer the reader to the monographs [20, 21, 22, 23, 24, 25, 26, 27].

The following easy-to-prove lemma is often used in the spectral theory of differential operators.

**Lemma 1.** Let the system  $\{x_n\}_{n \in \mathbb{N}}$  form a basis with parentheses for  $X$ . If the system  $\{x_n\}_{n \in \mathbb{N}}$  is uniformly minimal and the sequence  $\{n_{k+1} - n_k\}_{k \in \mathbb{N}}$  is bounded, then it forms a usual basis in  $X$ .

*Proof.* According to the criterion of basicity with parentheses, the system  $\{x_n\}_{n \in \mathbb{N}}$  is complete and minimal in  $X$ , and the projectors  $\{P_{n_k}\}_{k \in \mathbb{N}}$  are uniformly bounded, where  $\{x_n^*\}_{n \in \mathbb{N}} \subset X$  is the biorthogonal system for  $\{x_n\}_{n \in \mathbb{N}}$ )

$$P_{n_k} x = \sum_{n=1}^{n_k} \langle x, x_n^* \rangle x_n.$$

Therefore it suffices to show that the projectors  $\{P_n\}_{n \in \mathbb{N}}$  are uniformly bounded. Let  $n_k < n < n_{k+1}$ . Then

$$P_n x = P_{n_k} x + \sum_{i=n_k+1}^n \langle x, x_i^* \rangle x_i.$$

Hence, we have

$$\begin{aligned} \|P_n x\| &\leq \|P_{n_k} x\| + \left\| \sum_{i=n_k+1}^n \langle x, x_i^* \rangle x_i \right\| \leq M \|x\| + (n - n_k) \sup_{n_k \leq i \leq n} \|x_i^*\| \|x_i\| \|x\| \\ &\leq M \|x\| + (n_{k+1} - n_k) \sup_{i \in \mathbb{N}} \|x_i^*\| \|x_i\| \|x\|. \end{aligned}$$

By the assumption,

$$\sup_{k \in \mathbb{N}} (n_{k+1} - n_k) < +\infty, \quad \sup_{i \in \mathbb{N}} \|x_i^*\| \|x_i\| < +\infty.$$

Therefore,

$$\|P_n x\| \leq \text{const} \|x\|, \quad n \in \mathbb{N}.$$

□

We will also need the concept of an almost normed system.

A system  $\{x_n\}_{n \in \mathbb{N}} \subset X$  in a  $B$ -space  $X$  is called almost normed if

$$0 < \inf_n \|x_n\|_X \leq \sup_n \|x_n\|_X < +\infty.$$

For a uniformly minimal system  $\{x_n\}_{n \in \mathbb{N}}$ , the almost normed property is equivalent to the following condition:

$$\sup_{n \in \mathbb{N}} \{\|x_n\|; \|x_n^*\|\} < +\infty,$$

where the dual system  $\{x_n^*\}_{n \in \mathbb{N}}$  is also almost normed.

A system  $\{\varphi_n\}_{n \in \mathbb{N}} \subset X$  is called  $\omega$ -linearly independent in  $X$  if

$$\sum_{n=1}^{\infty} \lambda_n \varphi_n = 0 \quad \text{in } X$$

implies  $\lambda_n = 0$  for all  $n \in \mathbb{N}$ .

In the sequel, we will use the following construction and some obvious facts. Let the following direct sum

$$X = X_1 \oplus \dots \oplus X_m$$

hold, where  $X_i$ ,  $i = \overline{1, m}$ , are some  $B$ -spaces. For convenience, we represent the elements in the space  $X$  as a vector

$$x \in X \Leftrightarrow x = (x_1, x_2, \dots, x_m),$$

where  $x_k \in X_k$ ,  $k = \overline{1, m}$ . The norm in  $X$  is defined by the formula

$$\|x\|_X = \sqrt{\sum_{i=1}^m \|x_i\|_{X_i}^2}.$$

Then we have  $X^* = X_1^* \oplus \dots \oplus X_m^*$  (see [13]), and for  $f \in X^*$  and  $x \in X$  the identity

$$\langle x, f \rangle = \sum_{i=1}^m \langle x_i, f_i \rangle,$$

holds true, where  $f = (f_1, \dots, f_m)$  and

$$\|f\|_{X^*} = \sqrt{\sum_{i=1}^m \|f_i\|_{X_i^*}^2}.$$

From now on, we omit the subscripts in the notation for norms.

Suppose we are given some system  $\{u_{in}\}_{n \in \mathbb{N}} \subset X_i$  for every  $i = \overline{1, m}$ . We consider the following system in the space  $X$ :

$$u_{in}^0 = \left( \underbrace{0, \dots, u_{in}, \dots, 0}_n \right), \quad i = \overline{1, m}; \quad n \in \mathbb{N}.$$

There is a following obvious relationship between the basis properties of these two systems.

**Corollary 1.** *Each of the following properties, completeness, minimality, uniform minimality or  $\omega$ -linear independence, basicity of the system  $\{u_{in}\}_{n \in \mathbb{N}}$  in the space  $X_i$  for every  $i = \overline{1, m}$  is equivalent to the same property of the system  $\{u_{in}^0\}_{i=\overline{1, m}; n \in \mathbb{N}}$  in the space  $X$ .*

We will also use the concept of the space of coefficients. We define it as follows. Let  $\vec{x} \equiv \{x_n\}_{n \in \mathbb{N}} \subset X$  be a non-degenerate system in a  $B$ -space  $X$ , i.e.,  $x_n \neq 0$  for all  $n \in \mathbb{N}$ . We define

$$\mathcal{K}_{\vec{x}} \equiv \left\{ \{\lambda_n\}_{n \in \mathbb{N}} : \text{the series } \sum_{n=1}^{\infty} \lambda_n x_n \text{ converges in } X \right\}.$$

We introduce the norm in  $\mathcal{K}_{\vec{x}}$ :

$$\|\vec{\lambda}\|_{\mathcal{K}_{\vec{x}}} = \sup_m \left\| \sum_{n=1}^m \lambda_n x_n \right\|, \quad \text{where } \vec{\lambda} = \{\lambda_n\}_{n \in \mathbb{N}}.$$

The space  $\mathcal{K}_{\vec{x}}$  equipped by the usual summation and multiplication by complex number is a Banach space. More information about the above facts can be found in [22, 23, 24, 28].

### 3. COMPLETENESS AND MINIMALITY

Let the following direct sum

$$X = X_1 \oplus \dots \oplus X_m$$

hold true, where  $X_i$ ,  $i = \overline{1, m}$ , are some  $B$ -spaces, and let some system  $\{u_{in}\}_{n \in \mathbb{N}}$  be given in the space  $X_i$  for every  $i = \overline{1, m}$ . We consider the following system in the space  $X$ :

$$\omega_{in} = (a_{i1}u_{1n}; \dots; a_{im}u_{mn}), \quad i = \overline{1, m}; \quad n \in \mathbb{N}, \quad (3)$$

where  $a_{ij}$  are some numbers. Let

$$A = (a_{ij})_{i, j=\overline{1, m}}, \quad \Delta = \det A.$$

The following theorem is true.

**Theorem 1.** *Let the system  $\{u_{in}\}_{n \in \mathbb{N}}$  be complete (minimal) in the space  $X_i$ ,  $i = \overline{1, m}$ . If  $\Delta \neq 0$ , then the system  $\{\omega_{in}\}_{i=\overline{1, m}; n \in \mathbb{N}}$  is also complete (minimal) in the space  $X$ .*

*Proof.* Let the system  $\{u_{in}\}_{n \in \mathbb{N}}$  be complete in  $X_i$ ,  $i = \overline{1, m}$ . If for some  $\vartheta \in X^*$

$$\langle \omega_{in}, \vartheta \rangle = 0, \quad i = \overline{1, m}, n = 1, 2, \dots,$$

then it follows from the representation  $X^* = X_1^* \oplus \dots \oplus X_m^*$  and  $\vartheta = (\vartheta_1, \dots, \vartheta_m)$ ,  $\vartheta_i \in X_i^*$ ,  $i = \overline{1, m}$ , that

$$\sum_{j=1}^m a_{ij} \langle u_{jn}, \vartheta_j \rangle = 0, \quad i = \overline{1, m}. \quad (4)$$

As  $\Delta = \det(a_{ij}) \neq 0$ , the homogeneous system of linear equations (4) has only the trivial solution for every  $n \in \mathbb{N}$ :

$$\langle u_{jn}, \vartheta_j \rangle = 0, \quad j = \overline{1, m}, \quad n \in \mathbb{N}.$$

As  $\vartheta_j \in X_j^*$ , the completeness of the system  $\{u_{jn}\}_{n \in \mathbb{N}}$  in  $X_j$  implies  $\vartheta_j = 0$ ,  $j = \overline{1, m}$ , i.e.  $\vartheta = 0$ .

Now let  $\{u_{in}\}_{n \in \mathbb{N}}$  be minimal in  $X_i$ , and  $\{\vartheta_{in}\}_{n \in \mathbb{N}} \subset X_i^*$  be the biorthogonal system. We consider the following system in  $X^*$ :

$$\nu_{in} = (b_{1i}\vartheta_{1n}; b_{2i}\vartheta_{2n}; \dots; b_{mi}\vartheta_{mn}), \quad i = \overline{1, m}, \quad n \in \mathbb{N}, \quad (5)$$

where the numbers  $b_{ji}$  are the elements of the inverse matrix  $A^{-1}$ . We have

$$\langle \omega_{in}, \nu_{lk} \rangle = \sum_{j=1}^m \sum_{s=1}^m a_{ij} b_{sl} \langle u_{nj}, \vartheta_{ks} \rangle = \sum_{j=1}^m a_{ij} b_{jl} \delta_{nk} = \delta_{il} \delta_{nk}, \quad i, l, n, k \in \mathbb{N}.$$

The latter relation means that the system  $\{\nu_{in}\}_{i=\overline{1, m}; n \in \mathbb{N}}$  is biorthogonal for  $\{\omega_{in}\}_{i=\overline{1, m}; n \in \mathbb{N}}$ , i.e., the system  $\{\omega_{in}\}_{i=\overline{1, m}; n \in \mathbb{N}}$  is minimal.  $\square$

In case  $\Delta = 0$  we have the following theorem.

**Theorem 2.** *Let the system  $\{u_{in}\}_{n \in \mathbb{N}}$  be minimal in  $X_i$  for every  $i = \overline{1, m}$ . If  $\Delta = 0$ , then the system  $\{\omega_{in}\}_{i=\overline{1, m}; n \in \mathbb{N}}$  defined by (3) is not minimal in  $X$ .*

*Proof.* Let us show that the system  $\{\omega_{in}\}_{i=\overline{1, m}}$  is linearly dependent for every  $n \in \mathbb{N}$ . The identity  $\det(a_{ij}) = 0$  yields that there exist numbers  $c_i$ ,  $i = \overline{1, m}$ , not all equal to zero, such that

$$\sum_{i=1}^m a_{ij} c_i = 0, \quad j = \overline{1, m}.$$

Then

$$\sum_{i=1}^m c_i \omega_{in} = \sum_{i=1}^m c_i \sum_{j=1}^m a_{ij} u_{jn}^0 = \sum_{j=1}^m \left( \sum_{i=1}^m a_{ij} c_i \right) u_{jn}^0 = 0 \quad \text{for all } n \in \mathbb{N}.$$

Thus, in case  $\Delta = 0$  the system  $\{\omega_{in}\}_{i=\overline{1, m}; n \in \mathbb{N}}$  is linearly dependent and not minimal.  $\square$

The following theorem is also true.

**Theorem 3.** *Let the system  $\{u_{in}\}_{n \in \mathbb{N}}$  be complete and minimal in  $X_i$  for every  $i = \overline{1, m}$ . If  $\Delta = 0$ , then the system  $\{\omega_{in}\}_{i=\overline{1, m}; n \in \mathbb{N}}$  is not complete and not minimal in  $X$ .*

*Proof.* Nonminimality of the system  $\{\omega_{in}\}_{i=\overline{1, m}; n \in \mathbb{N}}$  in  $X$  is implied by the previous theorem. Let us show that it is not complete in  $X$ . The identity

$$\Delta = \det(a_{ij}) = 0$$

yields that there exist the numbers  $c_j$ ,  $j = \overline{1, m}$ , not all equal to zero, such that

$$\sum_{j=1}^m a_{ij} c_j = 0, \quad j = \overline{1, m}.$$

Let

$$u_{jn}^0 = \left( 0, \dots, \underbrace{u_{jn}, \dots}_j, 0 \right) \in X, \quad j = \overline{1, m}.$$

It is not difficult to see that the system  $\{u_{jn}^0\}_{j=\overline{1, m}; n \in \mathbb{N}}$  is complete and minimal in  $X$ , and its biorthogonal system is of the form:

$$\vartheta_{jn}^0 = (0, \dots, \vartheta_{jn}, \dots 0), \quad j = \overline{1, m}; \quad n \in \mathbb{N},$$

where  $\{\vartheta_{jn}\}_{n \in \mathbb{N}} \subset X_j^*$  is the system biorthogonal to  $\{u_{jn}\}_{n \in \mathbb{N}}$ . We consider the functional

$$\vartheta = \sum_{s=1}^m c_s \vartheta_{s1}^0.$$

It is clear that  $\vartheta \in X^*$  and  $\vartheta \neq 0$ . Let us show that the functional  $\vartheta$  vanishes at the system  $\{\omega_{in}\}$ . Indeed, for  $n = 1$  we have

$$\langle \omega_{i1}, \vartheta \rangle = \sum_{j=1}^m a_{ij} \langle u_{j1}^0, \vartheta \rangle = \sum_{j=1}^m a_{ij} \sum_{k=1}^m c_k \langle u_{j1}^0, \vartheta_{s1}^0 \rangle = \sum_{j=1}^m a_{ij} \sum_{s=1}^m c_s \delta_{js} = \sum_{j=1}^m a_{ij} c_j = 0.$$

For  $n \neq 1$  we have

$$\langle \omega_{in}, \vartheta \rangle = \sum_{j=1}^m a_{ij} \sum_{s=1}^m c_s \langle u_{jn}^0, \vartheta_{s1}^0 \rangle = 0.$$

Thus, the system  $\{\omega_{in}\}_{i=\overline{1,m}; n \in \mathbb{N}}$  is not complete in  $X$ . □

The minimality condition for the system  $\{u_{in}\}_{n \in \mathbb{N}}$  in the space  $X_i$  for every  $i = \overline{1,m}$  in Theorem 3 is significant, because the system  $\{\omega_{in}\}_{i=\overline{1,m}; n \in \mathbb{N}}$  can be complete in  $X$  only if the system  $\{u_{in}\}_{n \in \mathbb{N}}$  is complete in  $X_i$ ,  $i = \overline{1,m}$ , and  $\Delta = 0$ . Let us give an example of such a system. Let  $X_1 = X_2 = X$ , where  $X$  is some  $H$ -space with an orthonormal basis  $\{e_n\}_{n \in \mathbb{N}}$ . Let  $u_{in} = e_n$ ,  $n = 1, 2, \dots$ , and let the system  $\{u_{2n}\}_{n \in \mathbb{N}}$  be defined in  $X_2$  by the following expressions

$$u_{2;n+i} = e_i, \quad n = \frac{(k-1)k}{2}, \quad i = \overline{1,k}; \quad k \in \mathbb{N}.$$

We note that every element  $e_n$  appears in this system infinitely many times. We consider the following system in the space  $X = X_1 \oplus X_2$ :

$$\omega_{1n} = (u_{1n}; u_{2n}), \quad \omega_{2n} = (0; 0), \quad n \in \mathbb{N}.$$

Obviously, the matrix corresponding to this system is

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

and  $\Delta = \det A = 0$ . Let us show that the system  $\{\omega_{1n}\}_{n \in \mathbb{N}}$  is complete in  $X$ . Let  $\vartheta \in X$ ,  $\vartheta = (\vartheta_1; \vartheta_2)$ , and assume

$$\langle \omega_{1n}, \vartheta \rangle = (e_n, \vartheta_1) + (u_{2n}, \vartheta_2) = 0, \quad \forall n \in \mathbb{N}. \tag{6}$$

Definition of system  $\{u_{2n}\}$  combined with the relation (6) implies that for every  $n \in \mathbb{N}$  there exists a sequence of indices  $\{n_k\}$  such  $n_1 < n_2 < \dots < n_k \rightarrow \infty, k \rightarrow \infty$  and

$$(e_n, \vartheta_1) + (e_{n_k}, \vartheta_2) = 0, \quad n \in \mathbb{N}.$$

As  $(e_{n_k}, \vartheta_2) \rightarrow 0$  when  $k \rightarrow \infty$ , it follows that

$$(e_n, \vartheta_1) = 0, \quad \forall n \in \mathbb{N}.$$

Consequently, the completeness of the system  $\{e_n\}$  implies  $\vartheta_1 = 0$ . Then from (6) we obtain

$$(u_{2n}, \vartheta_2) = 0, \quad n \in \mathbb{N}.$$

Now it follows from the completeness of the system  $\{u_{2n}\}_{n \in \mathbb{N}}$  that  $\vartheta_2 = 0$ , and, as a result,  $\vartheta = 0$ . And in its turn, this means that the system  $\{\omega_{1n}\}_{n \in \mathbb{N}}$  is complete in  $X$ .

## 4. BASICITY

We suppose that the direct expansion  $X = X_1 \oplus \dots \oplus X_m$  holds true, where  $X_k$ ,  $k = \overline{1, m}$  are some  $B$ -spaces. Let  $T_{ij} : X_i \rightarrow X_j$  be some operators. We consider the system

$$\sum_{i=1}^m a_{ij} T_{ij} x_i = y_j, \quad j = \overline{1, m}, \quad (7)$$

where  $y_j \in X_j$ ,  $j = \overline{1, m}$ , are the given, and  $x_i \in X_i$ ,  $i = \overline{1, m}$ , are the elements to be determined. We assume that the spaces  $X_k$ ,  $k = \overline{1, m}$ , are pairwise isomorphic and  $T_{ij}$  performs a corresponding isomorphism. Besides, we assume that the following conditions are satisfied:

A)  $T_{ii} = I_i$ ,  $T_{ij} = T_{ji}^{-1}$ ,  $T_{jk} T_{ij} = T_{ik}$  for all  $i, j = \overline{1, m}$ , where  $I_i$  is the identity operator in  $X_i$ .

Applying the operator  $T_{j1} = T_{1j}^{-1}$  to the  $j^{\text{th}}$  equation in the system (7), we obtain the following system

$$\sum_{i=1}^m a_{ij} T_{i1} x_i = T_{j1} y_j, \quad j = \overline{1, m}.$$

Let  $\tilde{x}_i = T_{i1} x_i$ ,  $\tilde{y}_j = T_{j1} y_j$ . It is clear that  $\tilde{x}_i, \tilde{y}_j \in X_1$ . As a result, we obtain the following system of linear equations in the space  $X_1$ :

$$\sum_{i=1}^m a_{ij} \tilde{x}_i = \tilde{y}_j, \quad j = \overline{1, m}.$$

If the determinant of this system is non-zero,  $\Delta = \det(a_{ij}) \neq 0$ , then it is clear that this system is uniquely solvable with respect to  $\tilde{x}_i$ . Then the system (7) is also uniquely solvable.

Thus, the following lemma is true.

**Lemma 2.** *Let the operators  $T_{ij} : X_i \rightarrow X_j$  perform an isomorphism between  $X_i$  and  $X_j$ , the conditions A) be satisfied and  $\Delta \neq 0$ . Then the system (5) is uniquely solvable for each  $y \in X$ ,  $y = (y_1, \dots, y_m)^t$ , and moreover, there exists  $M > 0$ :*

$$\|x\|_X \leq M \|y\|_X, \quad (8)$$

where  $x = (x_1, \dots, x_m)$ .

Estimate (8) is implied immediately by the following representation for the solution of the system (7):

$$x_i = \sum_{j=1}^m b_{ij} T_{ji} y_j, \quad i = \overline{1, m},$$

where  $b_{ij}$  are the elements of the inverse matrix  $A^{-1}$ .

We consider the operator  $T : X \rightarrow X$  defined by the matrix  $(a_{ij} T_{ij})_{i,j=1}^m$ . Let all the conditions of Lemma 2 be satisfied. It follows from this lemma that  $\text{Ker } T = \{0\}$ ,  $R_T = X$ , and estimate (8) yields  $T \in L(X)$ . Then it follows from Banach theorem on the inverse operator that  $T$  is an automorphism in  $X$ . Hence, the following theorem is true.

**Theorem 4.** *Let  $T_{ij} \in L(X_i, X_j)$  be an isomorphism, the conditions A) be satisfied and  $\Delta \neq 0$ . Then the operator  $T : X \rightarrow X$  defined by the matrix  $(a_{ij} T_{ij})_{i,j=1}^m$  is an automorphism in  $X = X_1 \oplus \dots \oplus X_m$ .*

Suppose that the systems  $\{u_{in}\}_{n \in \mathbb{N}}$  form bases in the spaces  $X_i$ ,  $i = \overline{1, m}$ , respectively. Let these bases be isomorphic and  $T_{ij} \in L(X_i; X_j)$  perform a corresponding isomorphism

$$T_{ij} u_{in} = u_{jn}, \quad n \in \mathbb{N}.$$



It is clear that the spaces of coefficients  $\mathcal{K}_i$ ,  $i = \overline{1, m}$ , of these bases coincide. And vice versa, if  $\mathcal{K}_i$ ,  $i = \overline{1, m}$ , coincide, then there exist isomorphisms  $T_{ij} \in L(X_i; X_j)$  such that  $T_{ij}u_{in} = u_{jn}$ ,  $n \in \mathbb{N}$ . In addition, conditions A) are satisfied for the operators  $T_{ij}$ .

We consider the operator  $T : X \rightarrow X$  defined by the matrix  $(a_{ij}T_{ij})_{i,j=\overline{1,m}}$ , where  $\Delta = \det(a_{ij}) \neq 0$ . It is clear that the system

$$u_{in}^0 = (0, \dots, u_{in}, \dots, 0), \quad i = \overline{1, m}; \quad n \in \mathbb{N},$$

forms a basis in  $X$ . Then, obviously, the system

$$\omega_{in} = Tu_{in}^0 = (a_{i1}u_{1n}, a_{i2}u_{2n}, \dots, a_{im}u_{mn}),$$

also forms a basis in  $X$ . Hence, the following theorem is true.

**Theorem 5.** *Let the  $B$ -spaces  $X_i$ ,  $i = \overline{1, m}$ , be pairwise isomorphic and the systems  $\{u_{in}\}_{n \in \mathbb{N}}$  form isomorphic bases in them, respectively. If  $\Delta \neq 0$ , then the system  $\{\omega_{in}\}_{i=\overline{1,m}; n \in \mathbb{N}}$  forms a basis in  $X$  isomorphic to the basis  $\{u_{in}^0\}_{i=\overline{1,m}; n \in \mathbb{N}}$ .*

Now we consider the case when the isomorphism of subspaces  $X_i$ ,  $i = \overline{1, m}$ , is not required. As before, we assume that the expansion  $X = X_1 \oplus \dots \oplus X_m$  holds true. Let  $\{u_{in}\}_{n \in \mathbb{N}}$  form a basis in the space  $X_i$ ,  $i = \overline{1, m}$ , and the systems  $\{u_{in}^0\}_{i=\overline{1,m}; n \in \mathbb{N}}$ ,  $\{\omega_{in}\}_{i=\overline{1,m}; n \in \mathbb{N}}$ ,  $\{\vartheta_{in}^0\}_{i=\overline{1,m}; n \in \mathbb{N}}$ ,  $\{\nu_{in}\}_{i=\overline{1,m}; n \in \mathbb{N}}$  be defined as in Sections 2 and 3 and by expressions (3), (5). The following theorem is true.

**Theorem 6.** *If  $\Delta = \det(a_{ij}) \neq 0$ , then  $\{\omega_{in}\}_{i=\overline{1,m}; n \in \mathbb{N}}$  forms a basis with parentheses in the space  $X$ . If, in addition, the condition*

$$\sup_{i;n} \{\|u_{in}\|; \|\vartheta_{in}\|\} < +\infty,$$

*is satisfied, then the system  $\{\omega_{in}\}_{i=\overline{1,m}; n \in \mathbb{N}}$  forms a usual basis in  $X$ .*

*Proof.* We represent the system  $\{\omega_{in}\}$  in the form

$$u_{in} = \sum_{j=1}^m a_{ij}u_{jn}^0, \quad i = \overline{1, m}; \quad n \in \mathbb{N}. \quad (9)$$

As we showed in the previous section, the biorthogonal system is of the form

$$\nu_{in} = \sum_{j=1}^m b_{ji}\vartheta_{jn}^0, \quad i = \overline{1, m}; \quad n \in \mathbb{N}, \quad (10)$$

where the numbers  $b_{ji}$  are the elements of inverse matrix  $A^{-1}$ . It follows that

$$\begin{aligned} \sum_{i=1}^m \langle x, \nu_{in} \rangle \hat{u}_{in} &= \sum_{i=1}^m \sum_{j=1}^m \sum_{l=1}^m a_{ij}b_{li} \langle x, u_{ln}^0 \rangle u_{jn}^0 \\ &= \sum_{j=1}^m \sum_{l=1}^m \left( \sum_{i=1}^m b_{li}a_{ij} \right) \langle x, \vartheta_{ln}^0 \rangle u_{jn}^0 \\ &= \sum_{j=1}^m \sum_{l=1}^m \delta_{lj} \langle x, \vartheta_{ln}^0 \rangle u_{jn}^0 = \sum_{j=1}^m \langle x, \vartheta_{jn}^0 \rangle u_{jn}^0. \end{aligned}$$

Therefore,

$$S_N(x) = \sum_{n=1}^N \sum_{i=1}^m \langle x, \nu_{in} \rangle \omega_{in} = \sum_{n=1}^N \sum_{i=1}^m \langle x, \vartheta_{in}^0 \rangle u_{in}^0$$

$$= \sum_{i=1}^m \sum_{n=1}^N \langle x, \vartheta_{in}^0 \rangle u_{in}^0 \rightarrow x, \quad \text{as } N \rightarrow \infty.$$

Thus, the system  $\{\omega_{in}\}_{i=\overline{1,m}; n \in \mathbb{N}}$  forms a basis with parentheses in  $X$ .

Now we assume that the condition

$$\sup_{i,n} \{\|u_{in}\| ; \|\vartheta_{in}\|\} < +\infty$$

holds true. Then

$$\sup_{i,n} \{\|u_{in}^0\| ; \|\vartheta_{in}^0\|\} < +\infty,$$

and by representations (9) and (10) we obtain

$$\sup_{i,n} \{\|\omega_{in}\| ; \|\nu_{in}\|\} < +\infty$$

Hence, the system  $\{\omega_{in}\}$  is uniformly minimal and, by basicity with parenthesis criterion, forms a usual basis in  $X$ . □

**Remark 1.** Under the assumptions of Theorem 5, the spaces of coefficients of the systems  $\{\omega_{in}\}$  and  $\{u_{in}^0\}$  are isomorphic, and, using the matrix  $A$ , it is possible to construct explicitly the mapping that performs this isomorphism. In fact, let  $\mathcal{K}_1$  and  $\mathcal{K}_2$  be the spaces of coefficients corresponding to the systems  $\{u_{in}^0\}_{i=\overline{1,m}, n \in \mathbb{N}}$  and  $\{\omega_{in}\}_{i=\overline{1,m}; n \in \mathbb{N}}$ , and  $T_1$  and  $T_2$  be the natural isomorphisms, i.e.,

$$T_k : \mathcal{K}_k \leftrightarrow X, \quad k = 1, 2.$$

Let  $\vec{c} = (c_{in})_{i=\overline{1,m}; n \in \mathbb{N}} \in \mathcal{K}_1$ ,  $T_1 \vec{c} = x$  and  $\vec{d} = (d_{in})_{i=\overline{1,m}; n \in \mathbb{N}} \in \mathcal{K}_2$  and  $T_2 \vec{d} = x$ , where

$$x = \sum_{i=1}^m \sum_{n=1}^{\infty} c_{in} u_{in}^0, \quad c_{in} = \langle x, \vartheta_{in}^0 \rangle = \sum_{j=1}^m a_{ij} \langle x, \nu_{jn} \rangle = \sum_{j=1}^m a_{ij} d_{jn}.$$

On the other hand, we have

$$x = \sum_{n=1}^m \sum_{i=1}^{\infty} d_{in} \omega_{in},$$

with

$$d_{in} = \langle x, \nu_{in} \rangle = \sum_{j=1}^m b_{ji} \langle x, \vartheta_{in}^0 \rangle = \sum_{j=1}^m b_{ji} c_{jn},$$

where  $A = (a_{ij})$  and  $A^{-1} = (b_{ji})$ . Construct the operator  $A : \mathcal{K}_2 \rightarrow \mathcal{K}_1$  using the following infinite matrix

$$\mathcal{A} = \begin{pmatrix} A & 0 & 0 & \dots \\ 0 & A & 0 & \dots \\ 0 & 0 & A & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

There exists the inverse mapping  $\mathcal{A}^{-1} : \mathcal{K}_2 \rightarrow \mathcal{K}_1$  defined by the matrix

$$\mathcal{A}^{-1} = \begin{pmatrix} A^{-1} & 0 & 0 & \dots \\ 0 & A^{-1} & 0 & \dots \\ 0 & 0 & A^{-1} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

It is clear that for  $\vec{c} \in \mathcal{K}_1$  we have  $\mathcal{A}^{-1}\vec{c} = z$  and for  $\vec{d} \in \mathcal{K}_2$  we have  $\mathcal{A}\vec{d} = y$ , i.e.,  $D_A = \mathcal{K}_2$ ,  $R_A = \mathcal{K}_1$ . Besides, we have

$$\begin{aligned} \|T_1\vec{c}\| = \|x\| &= \left\| \sum_{i=1}^{\infty} \sum_{n=1}^m \langle x, \vartheta_{in}^0 \rangle u_{in}^0 \right\| \leq \sup_N \left\| \sum_{n=1}^N \sum_{i=1}^m \langle x, \vartheta_{in}^0 \rangle u_{in}^0 \right\| \\ &\leq \sup_N \left\| \sum_{n=1}^N \sum_{i=1}^m \langle x, \nu_{in} \rangle \omega_{in} \right\| \leq \|\vec{d}\|. \end{aligned}$$

It yields that

$$\|T_1 A \vec{d}\| \leq \|\vec{d}\|,$$

i.e. the operator  $T_1 \mathcal{A} = C$  is bounded. The boundedness of the operators  $T_1$ ,  $T_1^{-1}$  and  $C$  implies that  $\mathcal{A}$  is bounded. Then, by the Banach theorem on the inverse operator [13], the operator  $\mathcal{A}$  is boundedly invertible. Thus, the required isomorphism is constructed.

## 5. APPLICATION

Let us apply the obtained general results to the studying the basis properties (completeness, minimality, basicity, Riesz basicity) for the eigenfunctions of problem (1), (2) in  $L_p(-\pi, \pi)$ ,  $1 < p < +\infty$ . In order to do it, first we make the following construction.

We identify the spaces  $L_p(-\pi, 0)$  and  $L_p(0, \pi)$  with the corresponding subspaces  $L_p(-\pi, \pi)$ , and denote  $X_1 \equiv L_p(-\pi, 0)$ ,  $X_2 \equiv L_p(0, \pi)$ . We have  $X = X_1 + X_2$ . Thus, each element  $f \in L_p(-\pi, \pi)$ , is identified with the vector  $(f_1; f_2)$ , where  $f_1 = f|_{(-\pi, 0)}$ ,  $f_2 = f|_{(0, \pi)}$ ,  $f|_M$  is the restriction of  $f$  on  $M$ . We let  $u_{1n}^0 = (\sin nx; 0)$ ;  $u_{2n}^0 = (0; \sin nx)$ ,  $n \in N$ . It is clear that the systems  $\{u_{kn}^0\}_{n \in N}$ ,  $k = 1, 2$ , form isomorphic bases in  $X_k$ ,  $k = 1, 2$ , respectively. We have

$$u_n^{(1)} = u_{1n}^0 + u_{2n}^0, \quad u_n^{(2)} = u_{1n}^0 - u_{2n}^0 \quad \text{for all } n \in N.$$

Hence, in this case the identities  $a_{11} = 1$ ,  $a_{12} = 1$ ,  $a_{21} = 1$ ,  $a_{22} = -1$  are true and as a result,  $\det(a_{ij})_{i,j=1,2} \neq 0$ . Then Theorem 5 implies that the system  $\{u_n^{(1)}; u_n^{(2)}\}_{n \in N}$  forms a basis in  $L_p(-\pi, \pi)$ . Thus, the following corollary is true.

**Corollary 2.** *The system  $\{u_n^{(1)}; u_n^{(2)}\}_{n \in N}$  forms a basis in  $L_p(-\pi, \pi)$ ,  $1 < p < +\infty$ . Moreover, for the system of eigenfunctions  $\{u_n^{(1)}; \tilde{u}_n^{(2)}\}_{n \in N}$  of the problem (1), (2), the following properties are equivalent in  $L_p(-\pi, \pi)$ : i) complete; ii) minimal; iii) forms a basis; iv) as  $p = 2$ , forms a Riesz basis.*

*Proof.* Let us prove the validity of the second part of the corollary. Namely, let us show that the system  $\{u_n^{(1)}; u_n^{(2)}\}_{n \in N}$  satisfies Hausdorff-Young type inequality. We choose  $1 < p \leq 2$ ,  $f \in L_p(-\pi, \pi)$  and expand it w.r.t. this basis

$$f = \sum_{n=1}^{\infty} f_n^{(1)} u_n^{(1)} + \sum_{n=1}^{\infty} f_n^{(2)} u_n^{(2)},$$

where  $f_n^{(k)}$  are the biorthogonal coefficients of the function  $f$  w.r.t. system  $\{u_n^{(1)}; u_n^{(2)}\}_{n \in N}$ . The convergence of the series  $\sum_{n=1}^{\infty} f_n^{(k)} u_n^{(k)}$ ,  $k = 1, 2$ , in  $L_p(-\pi, \pi)$  implies immediately that the series

$$\sum_{n=1}^{\infty} f_n^{(k)} \sin nx, \quad k = 1, 2,$$

also converges in  $L_p(0, \pi)$ . Therefore, by the classical Hausdorff-Young theorem we obtain that

$$\| \{f_n^{(k)}\}_{n \in N} \|_{l_q} \leq c \left\| \sum_n f_n^{(k)} \sin nx \right\|_{L_p(0, \pi)},$$

where  $c > 0$  is a constant independent of  $f$ . Therefore,

$$\| \{f_n^{(1)}\} \|_{l_q} + \| \{f_n^{(2)}\} \|_{l_q} \leq c \left( \left\| \sum_n f_n^{(1)} \sin nx \right\|_{L_p(0, \pi)} + \left\| \sum_n f_n^{(2)} \sin nx \right\|_{L_p(0, \pi)} \right). \quad (11)$$

It is easy to see that

$$\left\| \sum_n f_n^{(k)} u_n^{(k)} \right\|_{L_p(-\pi, \pi)} \sim \left\| \sum_n f_n^{(k)} \sin nx \right\|_{L_p(0, \pi)},$$

i.e., there exists  $\delta > 0$  such that

$$\delta \left\| \sum_n f_n^{(k)} \sin nx \right\|_{L_p(0, \pi)} \leq \left\| \sum_n f_n^{(k)} u_n^{(k)} \right\|_{L_p(-\pi, \pi)} \leq \delta^{-1} \left\| \sum_n f_n^{(k)} \sin nx \right\|_{L_p(0, \pi)}. \quad (12)$$

We let  $Y_k = L \left[ \overline{\{u_n^{(k)}\}_{n \in N}} \right]$ ,  $k = 1, 2$ . The definition of the double basis implies immediately that there exists the direct decomposition  $X = Y_1 + Y_2$ . Hence, each  $f \in X$  has the unique representation  $f = f_1 + f_2$ ,  $f_k \in Y_k$ ,  $k = 1, 2$ . Then, as it is known (see e.g. [32]), there exists  $m > 0$  such that

$$\|f_1\|_{L_p(-\pi, \pi)} + \|f_2\|_{L_p(-\pi, \pi)} \leq m \|f\|_{L_p(-\pi, \pi)}.$$

In view of this relation, by (12) we obtain

$$\begin{aligned} c \left( \left\| \sum_n f_n^{(1)} \sin nx \right\|_{L_p(0, \pi)} + \left\| \sum_n f_n^{(2)} \sin nx \right\|_{L_p(0, \pi)} \right) &\leq \|f\|_{L_p(-\pi, \pi)} \leq \\ &\leq c^{-1} \left( \left\| \sum_n f_n^{(1)} \sin nx \right\|_{L_p(0, \pi)} + \left\| \sum_n f_n^{(2)} \sin nx \right\|_{L_p(0, \pi)} \right). \end{aligned}$$

As a result, by (11) we the following analogue of the Hausdorff-Young theorem

$$\| \{f_n^{(1)}\} \|_{l_q} + \| \{f_n^{(2)}\} \|_{l_q} \leq c \|f\|_{L_p(-\pi, \pi)}.$$

Thus, we have the following statement.

Let  $1 < p \leq 2$ . Then for each  $f \in L_p(-\pi, \pi)$  the inequality

$$\| \{f_n^{(1)}; f_n^{(2)}\} \|_{l_q} \leq c \|f\|_{L_p(-\pi, \pi)},$$

holds true, where  $\{f_n^{(1)}; f_n^{(2)}\}$  are the biorthogonal coefficients of  $f$  on the system  $\{u_n^{(1)}; u_n^{(2)}\}$ .

Now, let us consider the system of the eigenfunctions  $\{\tilde{u}_n^{(1)}; \tilde{u}_n^{(2)}\}$  of the problem (1),(2), where  $\tilde{u}_n^{(1)} = u_n^{(1)}$ ,  $\forall n \in N$ . Let  $\{f_n^{(1)}; f_n^{(2)}\}$  be an arbitrary finite set. We have

$$\begin{aligned} \Delta &= \left\| \sum_n f_n^{(1)} (\tilde{u}_n^{(1)} - u_n^{(1)}) + \sum_n f_n^{(2)} (\tilde{u}_n^{(2)} - u_n^{(2)}) \right\|_{L_p(-\pi, \pi)} \\ &\leq \sum_n \left( |f_n^{(1)}| \|\tilde{u}_n^{(1)} - u_n^{(1)}\|_{L_p(-\pi, \pi)} + |f_n^{(2)}| \|\tilde{u}_n^{(2)} - u_n^{(2)}\|_{L_p(-\pi, \pi)} \right) \\ &= \sum_n |f_n^{(2)}| \|\tilde{u}_n^{(2)} - u_n^{(2)}\|_{L_p(-\pi, \pi)}. \end{aligned}$$

Let  $1 < p \leq 2$ . We have

$$\Delta \leq \|\{f_n^{(2)}\}\|_{l_q} \left( \sum_n \|\tilde{u}_n^{(2)} - u_n^{(2)}\|_{L_p(-\pi, \pi)}^p \right)^{\frac{1}{p}} \leq A \left( \sum_n \|\tilde{u}_n^{(2)} - u_n^{(2)}\|_{L_p(-\pi, \pi)}^p \right)^{\frac{1}{p}} \|f\|_{L_p(-\pi, \pi)}.$$

Consider the case  $p > 2$ . In this case the inequality  $1 < q < 2$  is true and again by applying the Hausdorff-Young inequality we obtain

$$\Delta \leq \|\{f_n^{(2)}\}\|_{l_p} I_q \leq c I_q \|f\|_{L_q(-\pi, \pi)} \leq c I_q \|f\|_{L_p(-\pi, \pi)},$$

where

$$I_q = \left( \sum_n \|\tilde{u}_n^{(2)} - u_n^{(2)}\|_{L_p(-\pi, \pi)}^q \right)^{1/q}.$$

Further arguing is completely analogous to work [6].  $\square$

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