RECURSION OPERATOR FOR A SYSTEM WITH NON-RATIONAL LAX REPRESENTATION

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Abstract. We consider a hydrodynamic type system, waterbag model, that admits a dispersionless Lax representation with a logarithmic Lax function. Using the Lax representation, we construct a recursion operator of the system. We note that the constructed recursion operator is not compatible with the natural Hamiltonian representation of the system.

Keywords: recursion operator, hydrodynamic type systems, non-rational Lax representation.

Mathematics Subject Classification: 17B80, 37K10, 37K30, 70H06

1. Introduction

In the present paper we consider the so-called waterbag model [1],[2]. This hydrodynamic type system admits a dispersionless Lax representation with a logarithmic Lax function. Such systems have important applications in the topological field theories, see [3], [4] and the references therein. For a better understanding of such systems one needs to know a bi-Hamiltonian structure of a system and the corresponding recursion operator, see [5]-[7]. For the systems admitting dispersionless Lax representation the construction of bi-Hamiltonian structures and recursion operators is well understood in the case of a polynomial or rational Lax function [8]-[12]. The non-rational Lax functions present a much more difficult case. In the present paper we construct a recursion operator for the case of logarithmic Lax function. To our knowledge, in the literature, there are no other examples of recursion operators corresponding to a non-rational Lax function.

Let us give needed definitions. We introduce the algebra of Laurent series

$$\mathcal{A} = \left\{ \sum_{-\infty}^{\infty} u_i p^i : u_i \quad \text{are smooth functions decaying fast at infinity} \right\}, \tag{1.1}$$

with the Poisson bracket given by

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}. \tag{1.2}$$

Taking the Lax function

$$L = p - m\ln(p - c^1) + \ln(p - c^2) + \cdots + \ln(p - c^{m+1}) \tag{1.3}$$

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and using the Gel'fand-Dikii construction [13], we can write the hierarchy of integrable equations

$$L_{t_n} = \{(L^n)_{\geqslant 0}, L\} \qquad n = 1, 2, \dots . \tag{1.4}$$

The second equation of the hierarchy

$$L_t = \{(L^2)_{\geqslant 0}, L\} \tag{1.5}$$

leads to the waterbag model

$$c^j_t = \partial_x \left(\frac{(c^j)^2}{2} + mc^1 - c^2 - \dots - c^{m+1} \right), \tag{1.6}$$

where $j = 1, 2, \dots, (m+1)$. As we show, the above hierarchy admits the following recursion operator

$$\mathcal{R} = A\partial_x^{-1}, \tag{1.7}$$

where the matrix $A = (\gamma_{ij})$ has the entries

$$\gamma_{11} = c^1_x + \sum_{j=2}^{m+1} \frac{c^1_x - c^j_x}{c^1 - c^j}, \quad \gamma_{1k} = -\frac{c^1_x - c^k_x}{c^1 - c^k}, \quad \gamma_{k1} = m\frac{c^1_x - c^k_x}{c^1 - c^k},$$

$$\gamma_{kk} = c^k_x - m\frac{c^1_x - c^k_x}{c^1 - c^k} + \sum_{j=2, j\neq k}^{m+1} \frac{c^k_x - c^j_x}{c^k - c^j}, \qquad \gamma_{ki} = -\frac{c^k_x - c^i_x}{c^k - c^i}$$

$k \neq i$, and $k, i = 2, 3, \dots, m+1$.

We observe that the above system has an obvious Hamiltonian representation with the Hamiltonian operator $\mathcal{D} = J\partial_x$, where J is the matrix having one on the incidental diagonal and its other entries are zero.

In general, if a system has a bi-Hamiltonian representation with respect to a pair of Hamiltonian operators $\bar{\mathcal{D}}_1$ and $\bar{\mathcal{D}}_2$, one can construct a recursion operator $\bar{\mathcal{R}} = \bar{\mathcal{D}}_2\bar{\mathcal{D}}_1^{-1}$. Hence, one has $\bar{\mathcal{D}}_2 = \bar{\mathcal{R}}\bar{\mathcal{D}}_1$. For systems admitting dispersionless Lax representation one can generate the whole hierarchy of Hamiltonian operators $\bar{\mathcal{D}}_n = \bar{\mathcal{R}}^n\bar{\mathcal{D}}_1$ [14]. It turns out that in our case, if we apply the recursion operator $\mathcal{R}$ to the Hamiltonian operator $\mathcal{D}$, the resulting operator is not Hamiltonian. Thus, the recursion operator $\mathcal{R}$ and the Hamiltonian operator $\mathcal{D}$ are not compatible. For further studies, it is an interesting open question to find a bi-Hamiltonian representation of system (1.6).

The paper is organized as follows. In Section 2 we give a construction of the recursion operator of system (1.6) for general m. In Section 3 we give examples of system (1.6) and the corresponding recursion operator for $m = 1, 2, 3$.

2. Evaluation of recursion operator

Let us introduce new variables

$$c^1 = u \quad \text{and} \quad v^{j-1} = c^1 - c^j, \qquad j = 2, 3, \dots (m+1)\,. \tag{2.1}$$

In terms of the new variables, system (1.6) becomes

$$\begin{aligned} u_t &= uu_x + v^1_x + \dots v^n_x \\ v^1_t &= v^1u_x + (u - v^1)v^1_x \\ &\dots \\ v^m_t &= v^m u_x + (u - v^m)v^m_x \end{aligned} \tag{2.2}$$

System (2.2) admits a Lax representation

$$L_t = \{(L^2)_{\geqslant 1}, L\} \tag{2.3}$$

with Lax function

$$L = p + u + \ln\left(1 + \frac{v^1}{p}\right) + \ln\left(1 + \frac{v^2}{p}\right) + \cdots + \ln\left(1 + \frac{v^m}{p}\right). \tag{2.4}$$

Thus, we have the whole hierarchy of the symmetries for the system (2.2) given by

$$L_{t_n} = \{(L^n)_{\geqslant 1}, L\} \quad n = 1, 2, \dots \tag{2.5}$$

Let us construct a recursion operator for the above hierarchy of the symmetries. We construct the recursion operator by direct analysis of the Lax representation.

Let

$$L^n = a_n p^n + a_{n-1} p^{n-1} + \dots a_1 p + a_0 + a_{-1} p^{-1} + \dots \tag{2.6}$$

The next two lemmata provide some relations between coefficients of L^n and

$$L_{t_n} = u_{t_n} + \frac{v^1_{t_n}}{p + v^1} + \cdots + \frac{v^m_{t_n}}{p + v^m}.$$

.

Lemme 2.1. *For each $k = 2, 3 \dots m$ and each $n = 2, 3, \dots$ the identity*

$$\sum_{i=1}^{n} (-1)^{(i-1)} a_i (v^k)^i = \partial_x^{-1} v^k_{t_n} \tag{2.7}$$

holds true.

Proof. Using (2.6) we can write the equation (2.5) as

$$u_{t_n} + \frac{v^1_{t_n}}{p + v^1} + \cdots + \frac{v^m_{t_n}}{p + v^m} = (n a_n p^{n-1} + \cdots + 2a_2 p + a_1)\left(u_x + \frac{v^1_x}{p + v^1} + \cdots + \frac{v^m_x}{p + v^m}\right)$$
$$- (a_{n,x} p^n + \cdots + a_{2,x} p^2 + a_{1,x})\left(1 - \frac{v^1}{p(p + v^1)} - \cdots - \frac{v^m}{p(p + v^m)}\right)$$

Multiplying the above equation by $(p + v_1)(p + v_2) \dots (p + v_m)$ and then substituting $p = -v_k$, we obtain

$$v^k_{t_n} = \sum_{i=1}^{n} (-1)^{i-1} i a_i (v^k)^{i-1} v^k_x + \sum_{i=1}^{n} (-1)^{i-1} a_{i,x} (v^k)^i.$$

That is,

$$v^k_{t_n} = \left(\sum_{i=1}^{n} (-1)^{i-1} a_i (v^k)^i\right)_x.$$

□

Lemme 2.2. *For each $n = 2, 3, \dots,$ the identity $a_0 = \partial_x^{-1} u_{t_n}$ holds true.*

Proof. Lax equation (2.5) can be written as

$$L_{t_n} = -\{(L^n)_{\leqslant 0}, L\} \quad n = 1, 2, \dots$$

Using (2.6) and collecting coefficients of zero power of p in the above equations we have $u_{t_n} = a_{0,x}$. □

The above lemmata allow us to express the coefficients of $(L^{(n+1)}_{>0})_p$ and $(L^{(n+1)}_{>0})_x$ in terms of coefficients of $L^n_{\geqslant 0}$ and L_{t_n}.

Lemme 2.3. *Let*

$$\frac{1}{n+1}\left(L^{(n+1)}_{\geqslant 1}\right)_p = b_n p^{n-1} + \cdots + b_2 p + b_1. \tag{2.8}$$

Then

$$b_r = a_{r-1} + \sum_{k=1}^{m}\sum_{j=0}^{r-1}(v^k)^{-j}a_{r-j} + \sum_{k=1}^{m}(v^k)^{-r}\partial_x^{-1}v^k, \tag{2.9}$$

where $r = 1, 2, \dots n$.

Let

$$\frac{1}{n+1}\left(L^{(n+1)}_{\geqslant 1}\right)_x = d_n p^{n} + \cdots + d_2 p^2 + d_1 p. \tag{2.10}$$

Then

$$d_r = u_x a_r + \sum_{k=1}^{m}\sum_{j=0}^{r-1}(v^k)^{-j-1}v^k_x a_{r-j} + \sum_{k=1}^{m}(v^k)^{-r-1}v^k_x\partial_x^{-1}v^k, \tag{2.11}$$

where $r = 1, 2, \dots n$.

Proof. We have

$$\frac{1}{n+1}\left(L^{(n+1)}_{\geqslant 1}\right)_p = \left(L^{(n)}_{\geqslant 0}L_p\right)_{\geqslant 0}.$$

That is

$$\frac{1}{n+1}\left(L^{(n+1)}_{\geqslant 1}\right)_p = \left((a_n p^n + \cdots + a_0)\left(u_x + \sum_{k=1}^{m}\frac{v^k_x}{p+v_k}\right)\right)_{\geqslant 0}.$$

For each $k = 1, \dots m$, we expand $\dfrac{1}{p+v^k}$ as series in terms of p^{-1} at $p = \infty$ and multiply with $(a_n p^n + \cdots + a_0)$. Collecting coefficients at p^k, $k = 1, \dots m$, in the above identity and using Lemma 2.1, we obtain formula (2.9). The formula (2.11) can be obtained in the same way. □

Using the above lemmata, we find a recursion operator for the hierarchy (2.5).

Lemme 2.4. *The recursion operator for system (2.2) can be written as* $\mathcal{R} = C\partial_x^{-1}$, *where* C *is an* $(m+1)\times(m+1)$ *matrix. It is convenient to write matrix* C *as a sum of two matrices,* $C = (A + B)$. *Matrix* $A = (\alpha_{ij})$ *has the entries*

$$\begin{aligned}
&\alpha_{11} = u_x;\\
&\alpha_{1(j+1)} = v^j_x(v^j)^{-1}, && j = 1, 2, \dots, m;\\
&\alpha_{(j+1)1} = v^j_x, && j = 1, 2, \dots, m;\\
&\alpha_{(j+1)(j+1)} = (u_x - v^j_x), && j = 1, 2, \dots, m;\\
&\alpha_{(i+1)(j+1)} = 0, && i \neq j, \quad i, j = 1, 2, \dots, m;
\end{aligned}$$

Matrix $B = (\beta_{ij})$ *has the entries*

$$\begin{aligned}
&\beta_{11} = 0; \quad \beta_{1(j+1)} = 0, \quad \beta_{(j+1)1} = 0, \quad j = 1, 2, \dots, m;\\
&\beta_{(j+1)(j+1)} = \sum_{k=1, k\neq j}^{m}\frac{v^k_x - v^k(v^j)_x(v^j)^{-1}}{v^k - v^j}, \qquad j = 1, 2, \dots, m;\\
&\beta_{(i+1)(j+1)} = \frac{v^i_x - v^i v^j_x(v^j)^{-1}}{v^j - v^i}, \qquad i \neq j, \quad i, j = 1, 2, \dots, m.
\end{aligned}$$

Proof. Using notations of Lemma 2.3 the Lax equation (2.5) can be written as

$$u_{t_{n+1}}+\sum_{k=1}^{m}\frac{v^k_{t_{n+1}}}{p+v^k}=(n+1)(b_np^{n-1}+\cdots+b_2p+b_1)\left(u_x+\sum_{k=1}^{m}\frac{v^k_x}{p+v^k}\right)$$
$$(n+1)(d_np^n+\cdots+d_2p^2+d_1)\left(1-\sum_{k=1}^{m}\frac{v^k}{p(p+v^k)}\right). \tag{2.12}$$

We multiply the above equation by $(p+v^1)(p+v^2)\dots(p+v^m)$ and substitute the expressions for $b_i, d_i,\ i=1,2,\dots n$, given in Lemma 2.3. Equating coefficients at p^k, $k=1,2,\dots m$, we obtain a system of equations linear with respect to $v^k_{t_{n+1}}$, $k=1,2,\dots m$. Solving the system, we obtain the recursion operator given above. □

Remark 2.1. Let us a define vector $V=(u,v^1,v^2,\dots,v^m)$ and write system (2.2) as

$$V_t=K(V,V_x). \tag{2.13}$$

By straightforward calculations we check that the constructed above operator satisfies the criteria for recursion operators

$$\mathcal{R}_t=\mathbb{D}_K\mathcal{R}-\mathcal{R}\mathbb{D}_K, \tag{2.14}$$

where $\mathbb{D}_K$ is the Fréchet derivative of K.

Returning back to the original variables $c^1,\dots c^{m+1}$, we obtain recursion operator (1.7).

3. Examples

Let us consider some examples. We give examples in variables $c^1,c^2,\dots,c^{m+1}$.

Example 3.1. *Let us consider equation (1.6) with $m=1$. The equation becomes*

$$c^1_t=c^1c^1_x+c^1_x-c^2_x$$
$$c^2_t=c^2c^2_x+c^1_x-c^2_x$$

The above system admits the recursion operator

$$\begin{pmatrix} c^1_x+\dfrac{c^1_x-c^2_x}{c^1-c^2} & -\dfrac{c^1_x-c^2_x}{c^1-c^2} \\ \dfrac{c^1_x-c^2_x}{c^1-c^2} & c^2_x-\dfrac{c^1_x-c^2_x}{c^1-c^2}\end{pmatrix}\partial_x^{-1}.$$

Example 3.2. *Let us consider equation (1.6) with $m=2$. The equation becomes*

$$c^1_t=c^1c^1_x+2c^1_x-c^2_x-c^3_x$$
$$c^2_t=c^2c^2_x+2c^1_x-c^2_x-c^3_x$$
$$c^3_t=c^3c^3_x+2c^1_x-c^2_x-c^3_x$$

The above system admits the recursion operator

$$\begin{pmatrix} c^1_x+\dfrac{c^1_x-c^2_x}{c^1-c^2}+\dfrac{c^1_x-c^3_x}{c^1-c^3} & -\dfrac{c^1_x-c^2_x}{c^1-c^2} & -\dfrac{c^1_x-c^3_x}{c^1-c^3} \\ 2\dfrac{c^1_x-c^2_x}{c^1-c^2} & c^2_x-2\dfrac{c^1_x-c^2_x}{c^1-c^2}+\dfrac{c^2_x-c^3_x}{c^2-c^3} & -\dfrac{c^2_x-c^3_x}{c^2-c^3} \\ 2\dfrac{c^1_x-c^3_x}{c^1-c^3} & -\dfrac{c^3_x-c^2_x}{c^3-c^2} & c^3_x-2\dfrac{c^1_x-c^3_x}{c^1-c^3}+\dfrac{c^3_x-c^2_x}{c^3-c^2}\end{pmatrix}\partial_x^{-1}.$$

Example 3.3. *Let us consider equation (1.6) with $m = 3$. The equation becomes*

$$\begin{aligned} c^1_t &= c^1 c^1_x + 3c^1_x - c^2_x - c^3_x - c^4_x \\ c^2_t &= c^2 c^2_x + 3c^1_x - c^2_x - c^3_x - c^4_x \\ c^3_t &= c^3 c^3_x + 3c^1_x - c^2_x - c^3_x - c^4_x \\ c^4_t &= c^4 c^4_x + 3c^1_x - c^2_x - c^3_x - c^4_x \end{aligned}$$

The above system admits the recursion operator

$$\begin{pmatrix} c^1_x & -\frac{c^1_x - c^2_x}{c^1 - c^2} & -\frac{c^1_x - c^3_x}{c^1 - c^3} & -\frac{c^1_x - c^4_x}{c^1 - c^4} \\ 3\frac{c^1_x - c^2_x}{c^1 - c^2} & c^2_x - 3\frac{c^1_x - c^2_x}{c^1 - c^2} & -\frac{c^2_x - c^3_x}{c^2 - c^3} & -\frac{c^2_x - c^4_x}{c^2 - c^4} \\ 3\frac{c^1_x - c^3_x}{c^1 - c^3} & -\frac{c^3_x - c^2_x}{c^3 - c^2} & c^3_x - 3\frac{c^1_x - c^3_x}{c^1 - c^3} & -\frac{c^3_x - c^4_x}{c^3 - c^4} \\ 3\frac{c^1_x - c^4_x}{c^1 - c^4} & -\frac{c^4_x - c^2_x}{c^4 - c^2} & -\frac{c^4_x - c^3_x}{c^4 - c^3} & c^4_x - 3\frac{c^4_x - c^1_x}{c^4 - c^1} \end{pmatrix} \partial_x^{-1}$$

$$+ \begin{pmatrix} \sum_{j=2}^{4} \frac{c^1_x - c^j_x}{c^1 - c^j} & 0 & 0 & 0 \\ 0 & \sum_{j=2, j\neq 2}^{4} \frac{c^2_x - c^j_x}{c^2 - c^j} & 0 & 0 \\ 0 & 0 & \sum_{j=2, j\neq 3}^{4} \frac{c^3_x - c^j_x}{c^3 - c^j} & 0 \\ 0 & 0 & 0 & \sum_{j=2, j\neq 4}^{4} \frac{c^4_x - c^j_x}{c^4 - c^j} \end{pmatrix} \partial_x^{-1}$$

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