ISSN 2074-1863 Уфимский математический журнал. Том 6. № 2 (2014). С. 122-122.

УДК 517.55

LEVY'S PHENOMENON FOR ENTIRE FUNCTIONS OF SEVERAL VARIABLES

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Abstract. For entire functions $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, $z \in \mathbb{C}$, P. Lévy (1929) established that in the classical Wiman's inequality $M_f(r) \leqslant \mu_f(r)(\ln \mu_f(r))^{1/2+\varepsilon}$, $\varepsilon > 0$, which holds outside a set of finite logarithmic measure, the constant $1/2$ can be replaced almost surely in some sense by $1/4$; here $M_f(r) = \max\{|f(z)| \colon |z| = r\}$, $\mu_f(r) = \max\{|a_n|r^n \colon n \geq 0\}$, $r > 0$. In this paper we prove that the phenomenon discovered by P. Lévy holds also in the case of Wiman's inequality for entire functions of several variables, which gives an affirmative answer to the question of A. A. Goldberg and M. M. Sheremeta (1996) on the possibility of this phenomenon.

Keywords: Levy's phenomenon, random entire functions of several variables, Wiman's inequality

Mathematics Subject Classification: 30B20, 30D20

1. Introduction

For an entire function of the form

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

we denote $M_f(r) = \max\{|f(z)| \colon |z| = r\}$, $\mu_f(r) = \max\{|a_n|r^n \colon n \geq 0\}$, $r > 0$. It is well known ([1], [2]) that for each nonconstant entire function f and all $\varepsilon > 0$ the following inequality

$$M_f(r) \leqslant \mu_f(r)(\ln \mu_f(r))^{1/2+\varepsilon} \tag{1}$$

holds for $r > 1$ outside an exceptional set $E_f(\varepsilon)$ of finite logarithmic measure ($\int_{E_f(\varepsilon)} \frac{dr}{r} < +\infty$).

In this paper we consider entire functions of p complex variables

$$f(z) = f(z_1, \ldots, z_p) = \sum_{\|n\|=0}^{+\infty} a_n z^n, \tag{2}$$

where $z^n = z_1^{n_1} \ldots z_p^{n_p}$, $p \in \mathbb{N}$, $n = (n_1, \ldots, n_p) \in \mathbb{Z}_+^p$, $\|n\| = \sum_{j=1}^p n_j$. For $r = (r_1, \ldots, r_p) \in \mathbb{R}_+^p$ we denote

$$B(R) = \{t \in \mathbb{R}_+^p \colon t_j \geq R_j,\ j \in \{1, \ldots, p\}\},\ R = (R_1, \ldots, R_p),\ \ln_2 x = \ln\ln x,$$
$$r^\wedge = \min_{1 \leqslant i \leqslant p} r_i,\ M_f(r) = \max\{|f(z)| \colon |z_1| = r_1, \ldots, |z_p| = r_p\},$$
$$\mu_f(r) = \max\{|a_n|r_1^{n_1} \ldots r_p^{n_p} \colon n \in \mathbb{Z}_+^p\},\ \mathfrak{M}_f(r) = \sum_{\|n\|=0}^{+\infty} |a_n|r^n.$$

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Поступила 7 октября 2013г.

By Λ^p we denote the class of entire functions of form (2) such that $\frac{\partial}{\partial z_j} f(z) \not\equiv 0$ in $\mathbb{C}^p$ for any $j \in \{1, \dots, p\}$. We say that a subset E of $\mathbb{R}^p_+$ is a set of asymptotically finite logarithmic measure [9] if E is Lebesgue measurable in $\mathbb{R}^p_+$ and there exists an $R \in \mathbb{R}^p_+$ such that $E \cap B(R)$ is a set of finite logarithmic measure, i.e.

$$\int\limits_{E\cap B(R)}\cdots\int \prod_{j=1}^{p} \frac{dr_j}{r_j} < +\infty.$$

For entire functions of the form (2) analogues of inequality (1) are proved in [3, 5, 6, 9]. Also analogues of inequality (1) without exceptional sets for entire functions of several complex variables can be found in [10].

In particular, the following statement is proved in [9].

Theorem 1. *Let $f \in \Lambda^p$ and $\delta > 0$.*

a) *Then there exist $R \in \mathbb{R}^p_+$ and a subset E of $B(R)$ of finite logarithmic measure such that for all $r \in B(R)\backslash E$ we have*

$$\mathfrak{M}_f(r) \leqslant \mu_f(r)\Big(\prod_{i=1}^{p} \ln^{p-1} r_i \cdot \ln^p \mu_f(r)\Big)^{1/2+\delta}. \tag{3}$$

b) *If for some $\alpha \in \mathbb{R}^p_+$ we have $\mathfrak{M}(r) \geq \exp(r^\alpha) = \exp(r_1^{\alpha_1} \dots r_p^{\alpha_p})$, as $r^\wedge \to +\infty$ or more generally, for each $\beta > 0$*

$$\int\limits_{B(S)}\cdots\int \frac{\prod\limits_{i=1}^{p} dr_i}{r_1 r_2 \dots r_p \ln^\beta \mathfrak{M}_f(r)} < +\infty, \text{ as } S^\wedge \to +\infty, \tag{4}$$

then there exist $R \in \mathbb{R}^p_+$ and a subset E of $B(R)$ of finite logarithmic measure such that for all $r \in B(R)\backslash E$ we have

$$\mathfrak{M}_f(r) \leqslant \mu_f(r) \ln^{p/2+\delta} \mu_f(r).$$

2. Wiman's type inequality for random entire functions of several variables

Let $\Omega = [0, 1]$ and P be the Lebesgue measure on $\mathbb{R}$. We consider the Steinhaus probability space $(\Omega, \mathcal{A}, P)$ where $\mathcal{A}$ is the σ-algebra of Lebesgue measurable subsets of Ω. Let $X = (X_n(t))$ be some sequence of random variables defined in this space. For an entire function of the form $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ by $K(f, X)$ we denote the class of random entire functions of the form

$$f(z, t) = \sum_{n=0}^{+\infty} a_n X_n(t) z^n. \tag{5}$$

In the sequel, the notion "almost surely" will be used in the sense that the corresponding property holds almost everywhere with respect to Lebesgue measure P on $\Omega = [0, 1]$. We say that some relation holds almost surely in the class $K(f, X)$ if it holds for each entire function $f(z, t)$ of the form (5) almost surely in t.

In the case when $\mathcal{R} = (X_n(t))$ is the Rademacher sequence, i.e. $(X_n(t))$ is a sequence of independent uniformly distributed random variables on $[0, 1]$ such that $P\{t\colon X_n(t) = \pm 1\} = 1/2$, P. Levy [7] proved that for any entire function we can replace the constant $1/2$ by $1/4$ in the inequality (1) almost surely in the class $K(f, \mathcal{R})$. Later P. Erdős and A. Rényi [8] proved the same result for the class $K(f, H)$, where $H = (e^{2\pi i \omega_n(t)})$ is the Steinhaus sequence, i.e. $(\omega_n(t))$ is a sequence of independent uniformly distributed random variables on $[0, 1]$. This statement is true also for any class $K(f, X)$, where $X = (X_n(t))$ is multiplicative system (MS) uniformly bounded by the number 1. That is for all $n \in \mathbb{N}$ and $t \in [0,1]$ we have $|X_n(t)| \leqslant 1$ and

$$(\forall 1 \leqslant i_1 < i_2 < \cdots < i_k)\colon\ \mathbf{M}(X_{i_1}X_{i_2}\cdots X_{i_k}) = 0,$$

where $\mathbf{M}\xi$ is the expected value of a random variable ξ ([15]–[16]).

In the spring of 1996 during the report of P. V. Filevych at the Lviv seminar of the theory of analytic functions professors A. A. Goldberg and M. M. Sheremeta posed the following question (see [12]). Does Levy's effect take place for analogues of Wiman's inequality for entire functions of several complex variables?

In the papers [12]–[14] we have found an affirmative answer to this question for Fenton's inequality [4] for entire functions of two complex variables.

In this paper we will give answer to this question for Wiman's type inequality from [9] for entire functions of several complex variables.

The exceptional set in our statements is "smaller" than the exceptional set in the corresponding theorems from [4], [12]–[14]. The method of proof in this paper differs from the method of the papers [4], [12]–[14].

Let $Z = (Z_n(t))$ be a complex sequence of random variables $Z_n(t) = X_n(t) + iY_n(t)$ such that both $X = (X_n(t))$ and $Y = (Y_n(t))$ are real MS and $K(f, Z)$ the class of random entire functions of the form

$$f(z,t) = \sum_{\|n\|=0}^{+\infty} a_n Z_n(t) z_1^{n_1} \ldots z_p^{n_p}.$$

Theorem 2. *Let $Z = (Z_n(t))$ be a MS uniformly bounded by the number 1, $\delta > 0$, $f \in \Lambda^p$.*

a) *Then almost surely in $K(f, Z)$ there exist $R \in \mathbb{R}_+^p$ and a subset E^* of $B(R)$ of finite logarithmic measure such that for all $r \in B(R)\backslash E^*$ we have*

$$M_f(r,t) = \max_{|z|=r} |f(z,t)| \leqslant \mu_f(r)\Big(\ln^p \mu_f(r) \cdot \prod_{i=1}^{p} \ln^{p-1} r_i\Big)^{1/4+\delta}. \tag{6}$$

b) *If for some $\alpha \in \mathbb{R}_+^p$ we have*

$$\mathfrak{M}(r) \geq \exp(r^\alpha) = \exp(r_1^{\alpha_1} \ldots r_p^{\alpha_p}) \text{ as } r^\wedge \to +\infty$$

or more generally, for each $\beta > 0$ inequality (4) holds, then almost surely in $K(f, Z)$ there exist $R \in \mathbb{R}_+^p$ and a subset E of $B(R)$ of finite logarithmic measure such that for all $r \in B(R)\backslash E$ we get

$$M_f(r,t) \leqslant \mu_f(r) \ln^{p/4+\delta} \mu_f(r). \tag{7}$$

Lemma 1 ([10])**.** *Let $X = (X_n(t))$ be a MS uniformly bounded by the number 1. Then for each $\beta > 0$ there exists a constant $A_{\beta p} > 0$, which depends on p and β only such that for all $N \geq N_1(p) = \max\{p, 4\pi\}$ and $\{c_n\colon \|n\| \leqslant N\} \subset \mathbb{C}$ we have*

$$P\Bigg\{t\colon \max\Bigg\{\Bigg|\sum_{\|n\|=0}^{N} c_n X_n(t) e^{in_1\psi_1} \ldots e^{in_p\psi_p}\Bigg| \colon \psi \in [0,2\pi]^p\Bigg\} \geq \geq A_{\beta p} S_N \ln^{\frac{1}{2}} N\Bigg\} \leqslant \frac{1}{N^\beta} \tag{8}$$

where $S_N^2 = \sum_{\|n\|=0}^{N} |c_n|^2$.

By H we denote the class of function $h\colon \mathbb{R}_+^p \to \mathbb{R}_+$ such that

$$\int_1^{+\infty} \cdots \int_1^{+\infty} \frac{du_1 \ldots du_p}{h(u)} < +\infty.$$

We also define for all $i \in \{1, \dots, p\}$

$$\partial_i \ln \mathfrak{M}_f(r) = r_i \frac{\partial}{\partial r_i}(\ln \mathfrak{M}_f(r)) = \frac{1}{\mathfrak{M}_f(r)} \sum_{\|n\|=0}^{+\infty} n_i |a_n| r^n.$$

Lemma 2 ([9])**.** *Let $h \in H$. Then there exist $R \in \mathbb{R}^p_+$ and a subset E' of $B(R)$ of finite logarithmic measure such that for all $r \in B(R) \backslash E'$ and $s \in \{1, \dots, p\}$ we have*

$$\partial_s \ln \mathfrak{M}_f(r) \leqslant h(\ln r_1, \dots, \ln r_{s-1}, \ln \mathfrak{M}_f(r), \ln r_{s+1}, \dots, \ln r_p). \tag{9}$$

Proof of Theorem 2. Without loss of generality we may suppose that $Z = X = (X_n(t))$ is a MS. Indeed, if $Z_n(t) = X_n(t) + iY_n(t)$ then we obtain

$$f(z,t) = \sum_{\|n\|=0}^{+\infty} a_n X_n(t) z^n + \sum_{\|n\|=0}^{+\infty} i a_n Y_n(t) z^n = f_1(z,t) + f_2(z,t),$$

where $f_1, f_2 \in K(f, X)$, and

$$\max\{\mu(r, f_1(\cdot, t)), \mu(r, f_2(\cdot, t))\} \leqslant \mu(r, f) = \max\{|a_n| r_1^{n_1} \dots r_p^{n_p} \colon n \in \mathbb{Z}^p_+\}$$

for all $r \in \mathbb{R}^p_+$ and $t \in [0,1]$. Then from inequality (6) we obtain that there exists a set E_0 of asymptotically finite logarithmic measure such that for all $r \in B(R) \backslash E_0$ almost surely in $K(f, Z)$

$$M_{f_j}(r,t) \leqslant \mu_f(r) \Big(\prod_{i=1}^{p} \ln^{p-1} r_i \cdot \ln^p \mu_f(r)\Big)^{1/4+\delta_0}, \quad j \in \{1,2\},\ \delta_0 > 0.$$

So, for large enough $R^\wedge$ and for all $r \in B(R) \backslash E_0$ almost surely in $K(f, Z)$ we get

$$M_f(r,t) \leqslant M_{f_1}(r,t) + M_{f_2}(r,t) \leqslant$$

$$\leqslant 2\mu_f(r) \Big(\prod_{i=1}^{p} \ln^{p-1} r_i \cdot \ln^p \mu_f(r)\Big)^{1/4+\delta_0} < \mu_f(r) \Big(\prod_{i=1}^{p} \ln^{p-1} r_i \cdot \ln^p \mu_f(r)\Big)^{1/4+2\delta_0}.$$

For any $j \in \{1, \dots, p\}$ we have

$$\lim_{r_j \to +\infty} \mu_f(r_1^0, \dots, r_{j-1}^0, r_j, r_{j+1}^0, \dots, r_p^0) = +\infty \tag{10}$$

for fixed $r_i^0 > 0,\ i \in \{1, \dots, p\} \backslash \{j\}$. Indeed, if (10) does not hold, then there exists a constant $C > 0$ such that for all $r_j > r_j^*$ we have $\mu_f(r_1^0, \dots, r_{j-1}^0, r_j, r_{j+1}^0, \dots, r_p^0) < C < +\infty$. Hence, $\#\{n_j \geq 1 \colon a_n \neq 0\} = 0$ and $\frac{\partial}{\partial z_j} f(z) \equiv 0$ in $\mathbb{C}^p$. So, $f \notin \Lambda^p$, which gives a contradiction.

For $k \in \mathbb{N} \cup \{0\}$ we denote $G_k = \{r = (r_1, \dots, r_p) \in \mathbb{R}^p_+ \colon k \leqslant \ln \mu_f(r) < k+1\} \cap [1; +\infty)^p$. Then $G_k \neq \varnothing$ for $k \geq k_0$ and from (10) we deduce that for all k the set G_k is a bounded set. Let $G_k^+ = \bigcup_{j=k}^{+\infty} G_j$ and

$$h(r) = \prod_{i=1}^{p} r_i \ln^{1+\delta_1} r_i \in H,\ \delta_1 > 0.$$

By Lemma 2 there exist $R_j \in \mathbb{R}^p_+$ and a subset E_j of $B(R_j)$ of finite logarithmic measure such that for all $r \in B(R_j) \backslash E_j$ and $j \in \{1, \dots, p\}$ we have

$$\sum_{\|n\|=0}^{+\infty} n_i |a_n| r^n \leqslant \mathfrak{M}_f(r) h(\ln r_1, \dots, \ln r_{s-1}, \ln \mathfrak{M}_f(r), \ln r_{s+1}, \dots, \ln r_n) \leqslant$$

$$\leqslant \mathfrak{M}_f(r) \ln \mathfrak{M}_f(r) \ln_2^{1+\delta_1} \mathfrak{M}_f(r) \prod_{i=1,\ i\neq j}^{p} \ln r_i \ln_2^{1+\delta_1} r_i.$$

We can choose $R \in \mathbb{R}^p_+$ so that $B(R) \subset \Big(\bigcap_{j=1}^{p} B(R_j)\Big) \cap [e^{e^2}, +\infty)^p$.

Then for for large enough $R^\wedge$ and for all $r\in B(R)\backslash(\cup_{i=1}^p E_i)$ we obtain

$$\sum_{\|n\|=0}^{+\infty}\|n\||a_n|r^n\leqslant \mathfrak{M}_f(r)\ln\mathfrak{M}_f(r)\ln_2^{1+\delta_1}\mathfrak{M}_f(r)\sum_{j=1}^p\Bigg(\prod_{i=1,\ i\neq j}^p\ln r_i\ln_2^{1+\delta_1}r_i\Bigg)\leqslant$$
$$\leqslant p\cdot\mathfrak{M}_f(r)\ln^{1+\delta_1/2}\mathfrak{M}_f(r)\prod_{i=1}^p\ln r_i\ln_2^{1+\delta_1}r_i,$$

By Theorem 1 we get for large enough $R^\wedge$ and for all $r\in B(R)\backslash(\bigcup_{i=1}^p E_i)$

$$\sum_{\|n\|=0}^{+\infty}\|n\||a_n|r^n\leqslant p\mu_f(r)\Big(\prod_{i=1}^p\ln^{p-1}r_i\cdot\ln^p\mu_f(r)\Big)^{1/2+\delta_1}\times$$
$$\times\Big(\ln\mu_f(r)+\Big(\frac12+\delta_1\Big)\Big((p-1)\sum_{i=1}^p\ln_2 r_i+p\ln_2\mu_f(r)\Big)\Big)^{1+\delta_1/2}\prod_{i=1}^p\ln r_i\ln_2^{1+\delta_1}r_i\leqslant$$
$$\leqslant\mu_f(r)(\ln\mu_f(r))^{p/2+(p+1)\delta_1+1}\Big(\prod_{i=1}^p\ln r_i\Big)^{(p-1)(1/2+\delta_1)+1}\Big(\prod_{i=1}^p\ln_2 r_i\Big)^{2+3\delta_1/2},$$

because $a_1x_1+\dots+a_kx_k<x_1\cdot\ldots\cdot x_k$ for large enough $x^\wedge\geq 1$, $x=(x_1,\dots,x_k)$. Therefore as $\delta_2=(p+1)\delta_1$ for large enough $R^\wedge$ and for all $r\in B(R)\backslash(\bigcup_{i=1}^p E_i)$ we obtain

$$\sum_{\|n\|=0}^{+\infty}\|n\||a_n|r^n\leqslant\mu_f(r)\ln^{p/2+1+\delta_2}\mu_f(r)\prod_{i=1}^p\Big(\ln^p r_i\ln_2^2 r_i\Big)^{1+\delta_2}.$$

So,

$$\sum_{\|n\|\geq d}|a_n|r^n\leqslant\sum_{\|n\|\geq d}\frac{\|n\|}{d}|a_n|r^n=\frac1d\sum_{\|n\|\geq d}\|n\||a_n|r^n\leqslant$$
$$\leqslant\frac1d\mu_f(r)\ln^{p/2+1+\delta_2}\mu_f(r)\prod_{i=1}^p\Big(\ln^p r_i\ln_2^2 r_i\Big)^{1+\delta_2}=\mu_f(r),\tag{11}$$

where

$$d=d(r)=\ln^{p/2+1+\delta_2}\mu_f(r)\prod_{i=1}^p\Big(\ln^p r_i\ln_2^2 r_i\Big)^{1+\delta_2}.$$

Let $G_k^*=G_k\setminus E_{p+1}$, $E_{p+1}=\bigcup_{i=1}^p(E_i\cup E^*)\cup\Big(\bigcup_{i=1}^{k_0-1}G_i\Big)$. By I we denote the set of integers $k\geq k_0$ such that $G_k^*\neq\varnothing$. Then $\#I=+\infty$. For $k\in I$ we choose a sequence $r^{(k)}\in G_k^*$. Then for all $r\in G_k^*$ we get

$$\mu_f(r^{(k)})<e^{k+1}\leqslant e\mu_f(r),\quad \mu_f(r)<e^{k+1}<e\mu_f(r^{(k)}),\tag{12}$$

and also

$$\bigcup_{k\in I}G_k^*=\bigcup_{k\in I}G_k\setminus E_{p+1}=\bigcup_{k=1}^{+\infty}G_k\setminus E_{p+1}=[1;+\infty)^p\setminus E_{p+1}.$$

For $k\in I$ we denote $N_k=[2d_1(r^{(k)})]$, where

$$d_1(r)=\ln^{p/2+1+\delta_2}(e\mu_f(r))\prod_{i=1}^p\Big(\ln^p r_i\ln_2^2 r_i\Big)^{1+\delta_2},$$

and for $r \in G_k^*$

$$W_{N_k}(r,t) = \max\left\{\left|\sum_{\|n\|\leqslant N_k} a_n r_1^{n_1}\dots r_p^{n_p} e^{in_1\psi_1+\ldots+in_p\psi_p} X_n(t)\right| : \psi \in [0,2\pi]^p\right\}.$$

For a Lebesgue measurable set $G \subset G_k^*$ and for $k \in I$ we denote

$$\nu_k(G) = \frac{\mathrm{meas}_p(G)}{\mathrm{meas}_p(G_k^*)},$$

where meas_p denotes the Lebesgue measure on $\mathbb{R}^p$. Note that ν_k is a probability measure defined on the family of Lebesgue measurable subsets of G_k^*.

Let $\Omega = \bigcup_{k\in I} G_k^*$ and $I = \{k_j : j \ge 0\}$, where $k_j < k_{j+1}$, $j \ge 0$. Without loss of generality we may assume that $k_0 = 0$. Then $E_{p+1} = \bigcup_{i=1}^p (E_i \cup E^*)$. For Lebesgue measurable subsets G of Ω we denote

$$\nu(G) = \sum_{j=0}^{+\infty} \frac{1}{2^{k_j}}\Big(1-\Big(\frac12\Big)^{k_{j+1}-k_j}\Big)\cdot \nu_{k_{j+1}}(G \cap G_{k_{j+1}}^*). \tag{13}$$

We note that $\nu_{k_{j+1}}(G_{k_{j+1}}^*) = 1$, therefore

$$\nu(\Omega) = \sum_{j=0}^{+\infty} \frac{1}{2^{k_j}}\Big(1-\Big(\frac12\Big)^{k_{j+1}-k_j}\Big)\nu_{k_{j+1}}(G_{k_{j+1}}^*) = \sum_{j=0}^{+\infty}\sum_{s=k_j+1}^{k_{j+1}} \frac{1}{2^s} = \sum_{s=1}^{+\infty}\frac{1}{2^s} = 1.$$

Thus ν is a probability measure, which is defined on measurable subsets of Ω. On $[0,1]\times\Omega$ we define the probability measure $P_0 = P \otimes \nu$, which is a direct product of the probability measures P and ν. Now for $k \in I$ we define

$$F_k = \{(t,r) \in [0,1]\times\Omega \colon W_{N_k}(r,t) > A_1 S_{N_k}(r)\ln^{1/2} N_k\},$$
$$F_k(r) = \{t \in [0,1] \colon W_{N_k}(r,t) > A_1 S_{N_k}(r)\ln^{1/2} N_k\},$$

where $S_{N_k}^2(r) = \sum_{\|n\|=0}^{N_k} |a_n|^2 r^{2n}$ and A_p is the constant from Lemma 1 with $\beta = 1$. Using Fubini's theorem and Lemma 1 with $c_n = a_n r^n$ and $\beta = 1$, we get for $k \in I$

$$P_0(F_k) = \int\limits_{\Omega}\Bigg(\int\limits_{F_k(r)} dP\Bigg)d\nu = \int\limits_{\Omega} P(F_k(r))d\nu \leqslant \frac{1}{N_k}\nu(\Omega) = \frac{1}{N_k}.$$

Note that $N_k > \ln^{p/2+1}\mu_f(r^{(k)}) \ge k^{3/2}$. Therefore $\sum_{k\in I} P_0(F_k) \leqslant \sum_{k=1}^{+\infty} k^{-3/2} < +\infty$. By Borel-Cantelli's lemma the infinite quantity of the events $\{F_k \colon k \in I\}$ may occur with probability zero. So,

$$P_0(F) = 1, \quad F = \bigcup_{s=1}^{+\infty}\bigcap_{k\ge s, k\in I} \overline{F_k} \subset [0,1]\times\Omega.$$

Then for any point $(t,r) \in F$ there exists $k_0 = k_0(t,r)$ such that for all $k \ge k_0$, $k \in I$ we have

$$W_{N_k}(r,t) \leqslant A_1 S_{N_k}(r)\ln^{1/2} N_k. \tag{14}$$

Let P_j be a probability measure defined on $(\Omega_j, \mathcal{A}_j)$, where $\mathcal{A}_j$ is a σ-algebra of subsets Ω_j $(j \in \{1,\dots,p\})$ and P_0 is the direct product of probability measures $P_1,\dots,P_p$ defined on $(\Omega_1\times\ldots\times\Omega_p, \mathcal{A}_1\times\ldots\times\mathcal{A}_p)$. Here $\mathcal{A}_1\times\ldots\times\mathcal{A}_p$ is the σ-algebra, which contains all $A_1\times\ldots\times A_p$, where $A_j \in \mathcal{A}_j$. If $F \subset \mathcal{A}_1\times\ldots\times\mathcal{A}_p$ such that $P_0(F) = 1$, then in the case when projection

$$F_1 = \{t_1 \in \Omega_1 \colon (\exists (t_2,\dots,t_p) \in \Omega_2\times\ldots\times\Omega_p)[(t_1,\dots,t_p) \in F]\}$$

of the set F on Ω_1 is P_1-measurable we have $P_1(F_1) = 1$.

By F_Ω we denote the projection of F on Ω, i.e. $F_\Omega = \{r \in \Omega\colon (\exists t)[(t,r) \in F]\}$. Then $\nu(F_\Omega) = 1$. Similarly, the projection of F on $[0,1]$, $F_{[0,1]} = \bigcup_{r\in\Omega} F(r)$, we obtain $P(F_{[0,1]}) = 1$.
Let $F^\wedge(t) = \{r \in \Omega : (t,r) \in F\}$. By Fubini's theorem we have

$$0 = \int\limits_X (1-\chi_F)dP_0 = \int\limits_0^1 \Bigg(\int\limits_\Omega (1-\chi_{F^\wedge(t)})d\nu\Bigg)dP.$$

So P-almost everywhere $0 = \int_\Omega (1-\chi_{F^\wedge(t)})d\nu = 1-\nu(F^\wedge(t))$, i.e. $\exists\, F_1 \subset F_{[0,1]}$, $P(F_1) = 1$ such that for all $t \in F_1$ we get $\nu(F^\wedge(t)) = 1$.
Indeed, if for some $k \in I$, $k = k_{j+1}$ we obtain $\nu_k(F^\wedge(t)\cap G_k^*) = q < 1$, then

$$\nu(F^\wedge(t)) = \sum_{k\in I} \nu_k(F^\wedge(t)\cap G_k^*) \leqslant \sum_{s=0}^{+\infty} \frac{1}{2^{k_s}}\Big(1-\Big(\frac12\Big)^{k_{s+1}-k_s}\Big) -$$
$$-(1-q)\frac{1}{2^{k_j}}\Big(1-\Big(\frac12\Big)^{k_{j+1}-k_j}\Big) = 1-(1-q)\frac{1}{2^{k_j}}\Big(1-\Big(\frac12\Big)^{k_{j+1}-k_j}\Big) < 1.$$

For any $t \in F_1$ and $k \in I$ we choose a point $r_0^{(k)}(t) \in G_k^*$ such that

$$W_{N_k}(r_0^{(k)}(t),t) \geq \frac34 M_k(t),\ \ M_k(t) \stackrel{\text{def}}{=} \sup\{W_{N_k}(r,t) : r \in G_k^*\}.$$

Then from $\nu_k(F^\wedge(t)\cap G_k^*) = 1$ for all $k \in I$ it follows that there exists a point $r^{(k)}(t) \in G_k^*\cap F^\wedge(t)$ such that

$$|W_{N_k}(r_0^{(k)}(t),t) - W_{N_k}(r^{(k)}(t),t)| < \frac14 M_k(t)$$

or

$$\frac34 M_k(t) \leqslant W_{N_k}(r_0^{(k)}(t),t) \leqslant W_{N_k}(r^{(k)}(t),t) + \frac14 M_k(t).$$

Since $(t, r^{(k)}(t)) \in F$, from inequality (13) we obtain

$$\frac12 M_k(t) \leqslant W_{N_k}(r^{(k)}(t),t) \leqslant A_1 S_{N_k}(r^{(k)}(t)) \ln^{1/2} N_k. \tag{15}$$

Now for $r^{(k)} = r^{(k)}(t)$ we get

$$S_N^2(r^{(k)}) \leqslant \mu_f(r^{(k)})\mathfrak{M}_f(r^{(k)}) \leqslant \mu_f^2(r^{(k)})\Big(\prod_{i=1}^p \ln^{p-1} r_i^{(k)} \cdot \ln^p \mu_f(r^{(k)})\Big)^{1/2+\delta}.$$

So, for $t \in F_1$ and all $k \geq k_0(t)$, $k \in I$ we obtain

$$S_N(r^{(k)}) \leqslant \mu_f(r^{(k)})\Big(\prod_{i=1}^p \ln^{p-1} r_i^{(k)} \cdot \ln^p \mu_f(r^{(k)})\Big)^{1/4+\delta/2}. \tag{16}$$

It follows from (12) that $d_1(r^{(k)}) \geq d(r)$ for $r \in G_k^*$. Then for $t \in F_1$, $r \in F^\wedge(t)\cap G_k^*$, $k \in I,\ k \geq k_0(t)$ we get

$$M_f(r,t) \leqslant \sum_{\|n\|\geq 2d_1(r^{(k)})} |a_n|r^n + W_{N_k}(r,t) \leqslant \sum_{\|n\|\geq 2d(r)} |a_n|r^n + M_k(t).$$

Finally, from (11), (15), (16) for $t \in F_1$, $r \in F^\wedge(t) \cap G_k^*$, $k \in I$ and $k \geq k_0(t)$ we deduce

$$M_f(r^{(k)}, t) \leqslant \mu_f(r^{(k)}) + 2A_p S_{N_k}(r^{(k)}) \ln^{1/2} N_k \leqslant$$
$$\leqslant \mu_f(r^{(k)}) + 2A_p \mu_f(r^{(k)}) \Big(\prod_{i=1}^{p} \ln^{p-1} r_i^{(k)} \cdot \ln^p \mu_f(r^{(k)})\Big)^{1/4+\delta/2} \times$$
$$\times \Big((p/2 + 1 + \delta_2) \ln_2(e\mu_f(r^{(k)})) + (1+\delta_2) \sum_{i=1}^{p} (p \ln_2 r_i^{(k)} + 2 \ln_3 r_i^{(k)})\Big)^{1/2}.$$

Using inequality (12) we get for $t \in F_1$, $r \in F^\wedge(t) \cap G_k^*$, $k \in I$ and $k \geq k_0(t)$

$$M_f(r, t) \leqslant C\mu_f(r) \Big(\prod_{i=1}^{p} \ln^{p-1} r_i \cdot \ln^p \mu_f(r)\Big)^{1/4+3\delta_2/4}. \tag{17}$$

We choose $k_1 > k_0(t)$ such that for all $r \in G_{k_1}^+$ we have

$$C \leqslant \Big(\prod_{i=1}^{p} \ln^{p-1} r_i \cdot \ln^p \mu_f(r)\Big)^{\delta_2/4}. \tag{18}$$

Using (17) and (18) we get that inequality (6) holds almost surely ($t \in F_1$, $P(F_1) = 1$) for all

$$r \in \Big(\bigcup_{k \in I} (G_k^* \cap F^\wedge(t)) \cap G_{k_1}^+\Big) \backslash E^* =$$
$$= ([1, +\infty)^p \cap G_{k_1}^+) \setminus (E^* \cup G^* \cup E_{p+1}) = [1, +\infty)^p \setminus E_{p+2},$$

where

$$E_{p+2} = E_{p+1} \cup G^* \cup E^*, \;\; G^* = \bigcup_{k \in I} (G_k^* \setminus F^\wedge(t)).$$

It remains to remark that $\nu(G^*)$ defined in (13) satisfies $\nu(G^*) = \sum_{k \in I} (\nu_k(G_k^*) - \nu_k(F^\wedge(t))) = 0$. Then for all $k \in I$ we obtain

$$\nu_k(G_k^* \setminus F^\wedge(t)) = \frac{\operatorname{meas}_p(G_k^* \setminus F^\wedge(t))}{\operatorname{meas}_p(G_k^*)} = 0,$$
$$\operatorname{meas}_p(G_k^* \setminus F^\wedge(t)) = \int \cdots \int\limits_{G_k^* \setminus F^\wedge(t)} \frac{dr_1 \ldots dr_p}{r_1 \ldots r_p} = 0. \qquad \square$$

3. Some examples

In this section we prove that the exponent $p/4 + \delta$ in the inequality (7) cannot be replaced by a number smaller than $p/4$. It follows from such a statement.

Theorem 3. *For $f(z) = \exp\{\sum_{i=1}^{p} z_i\}$ almost surely in $K(f, H)$ for $r \in E$ we have*

$$M_f(r, t) \geq \frac{1}{(8p)^p} \mu_f(r) \ln^{p/4} \mu_f(r),$$

where E is a set of infinite asymptotically logarithmic measure and $H = \{e^{2\pi i \omega_n}\}$, $\{\omega_n\}$ is a sequence of independent random variables uniformly distributed on $[0, 1]$.

In order to prove this theorem we need such a result.

Theorem 4 ([17])**.** *For the entire function $g(z) = e^z$ almost surely in $K(g, H)$ we have*

$$\varliminf_{r \to +\infty} \frac{M_g(r, t)}{\mu_g(r) \ln^{1/4} \mu_g(r)} \geq \sqrt{\frac{\pi}{8}}. \tag{19}$$

Proof of Theorem 3. For the entire function $f(z)=\exp\{\sum_{i=1}^p z_i\}$ we have $\ln\mathfrak{M}_f(r)=\sum_{i=1}^p r_i$ and for each $\beta>0$ we get

$$\int\limits_{(1,+\infty)^p}\!\!\!\cdots\int\frac{dr_1\ldots dr_p}{r_1\ldots r_p(r_1+\ldots+r_p)^\beta}<+\infty.$$

Therefore the function $f(z)$ satisfies condition (4). From (19) we have for $r\in(r_0,+\infty)^p$

$$M_f(r,t)>\frac{1}{2^p}\mu_f(r)\prod_{i=1}^p\ln^{1/4}\mu_g(r_i).$$

Denote $\psi(r)=\ln\mu_g(r)$. Remark that

$$A_t=\{r\colon r_1=t; r_i\in(t_1,t_2)=(\psi^{-1}(\psi(r_1)/2),\psi^{-1}(2\psi(r_1)))\}\subset$$
$$\subset\Big\{r\colon \prod_{i=1}^p\psi(r_i)\ge\frac{1}{(4p)^p}\Big(\sum_{i=1}^p\psi(r_i)\Big)^p\Big\}.$$

Indeed, if $r\in A_t$ then for fixed r_1 we obtain

$$\prod_{i=1}^p\psi(r_i)=\psi(r_1)\prod_{i=2}^p\psi(r_i)>\psi(r_1)\prod_{i=2}^p\frac{\psi(r_1)}{2}=\frac{\psi^p(r_1)}{2^{p-1}}=$$
$$=\frac{1}{2^{p-1}(2p-1)^p}(\psi(r_1)+2\psi(r_1)+\ldots+2\psi(r_1))^p>\frac{1}{(4p)^p}\Big(\sum_{i=1}^p\psi(r_i)\Big)^p.$$

For $r\in A=\bigcup_{t=r_0}^{+\infty}A_t$ we get

$$M_f(r,t)>\frac{1}{2^p}\mu_f(r)\prod_{i=1}^p\ln^{1/4}\mu_g(r_i)>\mu_f(r)\frac{1}{(8p)^p}\Big(\sum_{i=1}^p\ln\mu_g(r_i)\Big)^{p/4}>$$
$$>\frac{1}{(8p)^p}\mu_f(r)\ln^{p/4}\mu_f(r).$$

It remains to prove that the set A has infinite asymptotically logarithmic measure. It is known [11] that $t<\psi^{-1}(t)<3t/2,\ t\to+\infty$. Therefore,

$$\mathrm{meas}_p(A)=\int\limits_{r_0}^{+\infty}\int\limits_{t_1}^{t_2}\cdots\int\limits_{t_1}^{t_2}\frac{dr_1\ldots dr_p}{r_1\ldots r_p}=\int\limits_{r_0}^{+\infty}\Big(\int\limits_{t_1}^{t_2}\frac{dr_2}{r_2}\Big)^{p-1}\frac{dr_1}{r_1}=$$
$$=\int\limits_{r_0}^{+\infty}\Big(\ln\psi^{-1}(2\psi(r_1))-\ln\psi^{-1}\Big(\frac{\psi(r_1)}{2}\Big)\Big)^{p-1}\frac{dr_1}{r_1}>$$
$$>\int\limits_{r_0}^{+\infty}\Big(\ln(2\psi(r_1))-\ln\Big(\frac{3\psi(r_1)}{4}\Big)\Big)^{p-1}\frac{dr_1}{r_1}=\ln^{p-1}\frac{8}{3}\cdot\int\limits_{r_0}^{+\infty}\frac{dr_1}{r_1}=+\infty.\qquad\square$$

Acknowledgments. Authors are grateful to the Referee for their valuable suggestions and comments.

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