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Linearization criteria for a system of two second-order ordinary differential equations

Yulia Yu Bagderina

Institute of Mathematics with Computer Centre of Russian Academy of Sciences, 112 Chernyshevsky str., Ufa 450008, Russia

E-mail: yulya@mail.rb.ru

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Abstract

For a system of second-order ordinary differential equations conditions of linearizability to the form $x'' = 0$ are well known. However, an arbitrary linear system need not be equivalent via an invertible point transformation to this simple form. We provide the criteria for a system of two second-order equations to be mapped to the linear system of the general form. Necessary and sufficient conditions for linearization by means of a point transformation are given in terms of coefficients of the system. These results are illustrated with a number of examples.

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1. Introduction

Systems of second-order ordinary differential equations (ODEs) are of great interest and widely applicable. They arise in theory of surfaces, mechanics (Euler–Lagrange equations, e.g., in many-body problems), mathematical physics as symmetry reductions of partial differential equations (PDEs), and in the context of integrable systems. Some of them are linear (free-particle system, harmonic oscillator with time-dependent frequency) and simple for integration. In applications, linear equations often occur in disguised forms; it is only after a change of variables that the underlying linear structure of a nonlinear differential equation is uncovered. There then arises the problem of transforming a given differential equation into a linear one, which is called a linearization problem.

Linearization problems can be treated as a particular case of an equivalence problem. Two systems of second-order ODEs

$$x'' = f_1(t, x, y, x', y'), \qquad y'' = f_2(t, x, y, x', y') \tag{1.1}$$

are equivalent if there exists an invertible change of variables

$$\bar{t} = \theta(t, x, y), \qquad \bar{x} = \varphi_1(t, x, y), \qquad \bar{y} = \varphi_2(t, x, y), \qquad \Delta = \frac{\partial(\theta, \varphi_1, \varphi_2)}{\partial(t, x, y)} \neq 0, \tag{1.2}$$

which transforms one system into another. As it was shown by Sophus Lie [1], many methods of solving differential equations are equivalent to mapping a given equation into another one with known properties (known solution). Linear systems are considered to be the simplest class of easily solvable equations. Therefore, it is of interest to provide general linearizability criteria for the systems of nonlinear ODEs.

Lie was the first to solve the linearization problem for a scalar second-order ODE [1, chapter 14] (see also [2, 3]). For a system of two second-order ODEs, the following result is well known [4–7].

Theorem 1.1. *The system of two second-order ODEs (1.1) is reducible via a point transformation (1.2) to the simplest equations*

$$\bar{x}'' = 0, \qquad \bar{y}'' = 0 \tag{1.3}$$

if and only if it has the form

$$\begin{aligned} x'' &= K_1 + 2L_1x' + 2M_1y' + P_1x'^2 + 2S_1x'y' + Q_1y'^2 + x'(V_1x'^2 + 2V_0x'y' + V_2y'^2),\\ y'' &= K_2 + 2L_2y' + 2M_2x' + P_2y'^2 + 2S_2x'y' + Q_2x'^2 + y'(V_1x'^2 + 2V_0x'y' + V_2y'^2), \end{aligned} \tag{1.4}$$

with the coefficients K_j, L_j, M_j, P_j, Q_j, S_j, V_0, V_j, $j = 1, 2$, depending on t, x, y, which satisfy 15 relations:

$$\begin{array}{llllll} a_0 = 0, & a_1 = 0, & a_2 = 0, & a_3 = 0, & a_7 = 0, & \\ b_0 = 0, & b_1 = 0, & b_2 = 0, & b_3 = 0, & b_7 = 0, & a_8 - b_8 = 0, \\ a_4 - 2a_5 = 0, & 3a_5 - a_6 = 0, & b_4 - 2b_5 = 0, & 3b_5 - b_6 = 0, & & \end{array} \tag{1.5}$$

where

$$\begin{aligned} a_0 &= V_{0x} - V_{1y} - S_1V_1 + (P_1 - S_2)V_0 + Q_2V_2,\\ a_1 &= V_{2t} + Q_{1x} - S_{1y} + Q_1(S_2 - P_1) + S_1(S_1 - P_2) + (L_1 + L_2)V_2,\\ a_2 &= V_{0t} + S_{1x} - P_{1y} + Q_1Q_2 - S_1S_2 - 2M_1V_1 + (3L_1 - L_2)V_0 + 2M_2V_2,\\ a_3 &= Q_{1t} - M_{1y} + M_1(S_1 - P_2) + (L_2 - L_1)Q_1 + K_1V_2,\\ a_4 &= S_{1t} - M_{1x} + M_1(P_1 - S_2) + (L_2 - L_1)S_1 + K_1V_0,\\ a_5 &= S_{1t} - L_{1y} - M_1S_2 + M_2Q_1 + \tfrac{3}{2}K_1V_0 + \tfrac{1}{2}K_2V_2,\\ a_6 &= P_{2t} - L_{2y} + M_1S_2 - M_2Q_1 + \tfrac{1}{2}K_1V_0 + \tfrac{3}{2}K_2V_2,\\ a_7 &= M_{1t} - K_{1y} - (L_1 + L_2)M_1 + K_1S_1 + K_2Q_1,\\ a_8 &= L_{1t} - K_{1x} - L_1^2 - M_1M_2 + K_1P_1 + K_2S_1,\\ b_0 &= V_{0y} - V_{2x} - S_2V_2 + (P_2 - S_1)V_0 + Q_1V_1,\\ b_1 &= V_{1t} + Q_{2y} - S_{2x} + Q_2(S_1 - P_2) + S_2(S_2 - P_1) + (L_1 + L_2)V_1,\\ b_2 &= V_{0t} + S_{2y} - P_{2x} + Q_1Q_2 - S_1S_2 - 2M_2V_2 + (3L_2 - L_1)V_0 + 2M_1V_1,\\ b_3 &= Q_{2t} - M_{2x} + M_2(S_2 - P_1) + (L_1 - L_2)Q_2 + K_2V_1,\\ b_4 &= S_{2t} - M_{2y} + M_2(P_2 - S_1) + (L_1 - L_2)S_2 + K_2V_0,\\ b_5 &= S_{2t} - L_{2x} - M_2S_1 + M_1Q_2 + \tfrac{3}{2}K_2V_0 + \tfrac{1}{2}K_1V_1,\\ b_6 &= P_{1t} - L_{1x} + M_2S_1 - M_1Q_2 + \tfrac{1}{2}K_2V_0 + \tfrac{3}{2}K_1V_1,\\ b_7 &= M_{2t} - K_{2x} - (L_1 + L_2)M_2 + K_2S_2 + K_1Q_2,\\ b_8 &= L_{2t} - K_{2y} - L_2^2 - M_1M_2 + K_2P_2 + K_1S_2 \end{aligned} \tag{1.6}$$

are the first-order relative invariants of the system (1.4).

However, unlike a single equation, an arbitrary linear system may be irreducible by a point transformation to the simplest form (1.3). One of possible reasons is that it can possess 5 to 8 or 15 Lie point symmetries [8], the maximum number being achieved by equations (1.3) [9]. Moreover, (1.3) is the unique system of two second-order ODEs (up to the change of variables (1.2)) which admits a point symmetry group of maximal dimension [4]. Hence, it remains an open problem to describe all nonlinear systems of two second-order ODEs reducible by an invertible change of variables (1.2) to the arbitrary linear system.

Such general linearization criteria based on symmetry properties of the system (1.1) have been obtained in [10]. They are stated as follows.

Theorem 1.2. *System (1.1) is linearizable by means of a point transformation if and only if it admits an Abelian Lie algebra $L_{4.1}$ with the basis*

$$X_i = \xi_i^0(t, x, y)\partial_t + \xi_i^1(t, x, y)\partial_x + \xi_i^2(t, x, y)\partial_y, \qquad i = 1, \dots, 4,$$

such that $\mathrm{rank}\|\xi_i^k\| = 2$.

Theorem 1.3. *System (1.1) is linearizable by means of a point transformation if and only if it admits a four-dimensional Lie algebra $L_{4.2}$ with commutators $[X_i,\ X_j] = 0$, $[X_i,\ X_4] = X_i$, $i, j = 1, 2, 3$.*

These two theorems require calculating Lie symmetry algebra for equations (1.1), which implies integrating a system of linear PDEs in functions ξ^0, ξ^1, ξ^2. Note also the paper [11] where a new method of linearization for a system of two second-order ODEs is proposed. First given the system (1.1) is reduced to a single equation in $y(x)$ and then Lie's theorem on linearization of scalar second-order ODE is applied.

In this paper we provide general linearization criteria which, similarly to theorem 1.1, solve the problem in terms of the coefficients of the system (1.4). As is noted in [11], if the system (1.1) is linearizable, then it must be of the form (1.4) that is not changed by any transformation (1.2). The outline of this paper is as follows. In section 2 the case of transformation (1.2) with $\theta = \theta(t)$ is studied. The case of arbitrary transformation (1.2) (with $\theta_x \neq 0$ or $\theta_y \neq 0$) is considered in section 3. In section 4 our results are illustrated by examples of a system of geodesic equations, a system of interacting particles and the stationary reduction of the Heisenberg model and others.

2. Linearization by means of a restricted class of point transformations

First we solve the linearization problem when the function θ in (1.2) depends on t only.

Theorem 2.1. *The system of two second-order ODEs is reducible via a point transformation:*

$$\bar t = \theta(t), \qquad \bar x = \varphi_1(t, x, y), \qquad \bar y = \varphi_2(t, x, y), \qquad \Delta = \varphi_{1,x}\varphi_{2,y} - \varphi_{1,y}\varphi_{2,x} \neq 0, \tag{2.1}$$

to a linear system

$$\begin{aligned} \bar x'' &= p_1(\bar t)\bar x' + q_1(\bar t)\bar y' + l_1(\bar t)\bar x + m_1(\bar t)\bar y, \\ \bar y'' &= p_2(\bar t)\bar y' + q_2(\bar t)\bar x' + l_2(\bar t)\bar y + m_2(\bar t)\bar x \end{aligned} \tag{2.2}$$

if and only if it has the form

$$\begin{aligned} x'' &= K_1 + 2L_1x' + 2M_1y' + P_1x'^2 + 2S_1x'y' + Q_1y'^2, \\ y'' &= K_2 + 2L_2y' + 2M_2x' + P_2y'^2 + 2S_2x'y' + Q_2x'^2, \end{aligned} \tag{2.3}$$

with the coefficients K_j, L_j, M_j, P_j, S_j, Q_j, $j = 1, 2$, depending on t, x, y, which satisfy 20 relations:

$$a_j = 0, \qquad b_j = 0, \qquad j = 1, \ldots, 6, \tag{2.4}$$

$$\begin{array}{ll} a_{7x} + (S_2 - P_1)a_7 + S_1(a_8 - b_8) = 0, & a_{8x} + Q_2 a_7 - S_1 b_7 = 0, \\ a_{7y} + (P_2 - S_1)a_7 + Q_1(a_8 - b_8) = 0, & a_{8y} + S_2 a_7 - Q_1 b_7 = 0, \\ b_{7x} + (P_1 - S_2)b_7 + Q_2(b_8 - a_8) = 0, & b_{8x} - Q_2 a_7 + S_1 b_7 = 0, \\ b_{7y} + (S_1 - P_2)b_7 + S_2(b_8 - a_8) = 0, & b_{8y} - S_2 a_7 + Q_1 b_7 = 0. \end{array} \tag{2.5}$$

Proof. Substitution of (2.1) into a linear system (2.2) yields the equalities

$$\begin{aligned} x''\varphi_{j,x} &+ y''\varphi_{j,y} + \varphi_{j,tt} + 2x'\varphi_{j,tx} + 2y'\varphi_{j,ty} + x'^2\varphi_{j,xx} + 2x'y'\varphi_{j,xy} + y'^2\varphi_{j,yy} \\ &+(p_j\theta' + \theta''/\theta')(\varphi_{j,t} + x'\varphi_{j,x} + y'\varphi_{j,y}) + (l_j\varphi_j + m_j\varphi_{3-j})\theta'^2 \\ &+q_j(\varphi_{3-j,t} + x'\varphi_{3-j,x} + y'\varphi_{3-j,y}), \qquad j = 1, 2. \end{aligned}$$

Solving them for x'', y'' one obtains the system of second-order ODEs with the same dependence on x', y' as in equations (2.3). Equating the coefficients of like powers of x', y' in this system to the corresponding coefficients of equations (2.3) one can find expressions

$$\begin{aligned} \varphi_{j,tt} &= (p_j\theta' + \theta''/\theta')\varphi_{j,t} + q_j\theta'\varphi_{3-j,t} - K_1\varphi_{j,x} - K_2\varphi_{j,y} + (l_j\varphi_j + m_j\varphi_{3-j})\theta'^2, \\ \varphi_{j,tx} &= \tfrac{1}{2}(p_j\theta' + \theta''/\theta')\varphi_{j,x} + \tfrac{1}{2}q_j\theta'\varphi_{3-j,x} - L_1\varphi_{j,x} - M_2\varphi_{j,y}, \\ \varphi_{j,ty} &= \tfrac{1}{2}(p_j\theta' + \theta''/\theta')\varphi_{j,y} + \tfrac{1}{2}q_j\theta'\varphi_{3-j,y} - M_1\varphi_{j,x} - L_2\varphi_{j,y}, \\ \varphi_{j,xx} &= -P_1\varphi_{j,x} - Q_2\varphi_{j,y}, \qquad \varphi_{j,xy} = -S_1\varphi_{j,x} - S_2\varphi_{j,y}, \\ \varphi_{j,yy} &= -Q_1\varphi_{j,x} - P_2\varphi_{j,y}, \qquad j = 1, 2 \end{aligned} \tag{2.6}$$

for all second-order derivatives of the functions φ_1, φ_2. Therefore, the system of second-order ODEs (1.1) is reducible by a change of variables (2.1) to a linear system (2.2) if and only if it has the form (2.3) and the system of linear PDEs (2.6) is compatible.

In order to find the compatibility conditions we differentiate (2.6) with respect to t, x, y and compare mixed derivatives of the third order of the functions φ_1, φ_2. This leads to 16 relations:

$$\begin{array}{lll} a_1\varphi_{j,x} - b_2\varphi_{j,y} = 0, & a_2\varphi_{j,x} - b_1\varphi_{j,y} = 0, & \\ a_3\varphi_{j,x} + a_6\varphi_{j,y} = 0, & b_6\varphi_{j,x} + b_3\varphi_{j,y} = 0, & \\ a_4\varphi_{j,x} + b_5\varphi_{j,y} = 0, & a_5\varphi_{j,x} + b_4\varphi_{j,y} = 0, & j = 1, 2. \end{array} \tag{2.7}$$

$$\begin{aligned} &(a_8 - H/2 + (-1)^j\alpha_0\theta'^2)\varphi_{j,x} + b_7\varphi_{j,y} - \alpha_j\theta'^2\varphi_{3-j,x} = 0, \\ &a_7\varphi_{j,x} + (b_8 - H/2 + (-1)^j\alpha_0\theta'^2)\varphi_{j,y} - \alpha_j\theta'^2\varphi_{3-j,y} = 0, \qquad j = 1, 2, \end{aligned} \tag{2.8}$$

where $H = \theta'''/\theta' - \frac{3}{2}\theta''^2/\theta'^2 + \alpha_3\theta'^2$ and

$$\begin{aligned} \alpha_0 &= \tfrac{1}{4}(p_1' - p_2') + \tfrac{1}{2}(l_2 - l_1) + \tfrac{1}{8}\left(p_2^2 - p_1^2\right), \\ \alpha_j &= \tfrac{1}{2}q_j' - m_j - \tfrac{1}{4}(p_1 + p_2)q_j, \qquad j = 1, 2, \\ \alpha_3 &= \tfrac{1}{2}(p_1' + p_2' - q_1q_2) - l_1 - l_2 - \tfrac{1}{4}\left(p_1^2 + p_2^2\right). \end{aligned} \tag{2.9}$$

Note that from equations (2.8) formula

$$a_8 + b_8 = H \tag{2.10}$$

follows. Equations (2.7) represent six algebraic linear homogeneous systems for $a_1, \ldots, a_6$, $b_1, \ldots, b_6$ with nonzero determinant Δ. They have trivial solution given by (2.4). Differentiating equalities (2.8) with respect to x, y leads to four systems

$$\begin{aligned}
&(a_{7x} + (S_2 - P_1)a_7 + S_1(a_8 - b_8))\varphi_{j,x} + (b_{8x} - Q_2a_7 + S_1b_7)\varphi_{j,y} = 0,\\
&(b_{7x} + (P_1 - S_2)b_7 + Q_2(b_8 - a_8))\varphi_{j,y} + (a_{8x} + Q_2a_7 - S_1b_7)\varphi_{j,x} = 0,\\
&(a_{7y} + (P_2 - S_1)a_7 + Q_1(a_8 - b_8))\varphi_{j,x} + (b_{8y} - S_2a_7 + Q_1b_7)\varphi_{j,y} = 0,\\
&(b_{7y} + (S_1 - P_2)b_7 + S_2(b_8 - a_8))\varphi_{j,y} + (a_{8y} + S_2a_7 - Q_1b_7)\varphi_{j,x} = 0, \qquad j = 1, 2.
\end{aligned}$$

Their solution is trivial too and yields relations (2.5). Thus, we have obtained compatibility conditions (2.4) and (2.5) for the system (2.6) and this completes the proof. □

3. Linearization by point transformation of the general form

The main result of this paper is stated as follows.

Theorem 3.1. *The system of two second-order ODEs (1.1) is linearizable by a point transformation (1.2) with $\theta_x \neq 0$ if and only if it has the form (1.4) with the coefficients satisfying the conditions: (1) either*

$$\operatorname{rank} B = 0, \tag{3.1}$$

where B is 6×3-dimensional matrix with elements

$$\begin{aligned}
&B_{11} = -a_0, && B_{12} = b_0, && B_{13} = \tfrac14(b_2 - a_2),\\
&B_{21} = b_1, && B_{22} = \tfrac14(a_2 + 3b_2), && B_{23} = \tfrac14(b_6 + 5b_5) - b_4,\\
&B_{31} = \tfrac14(3a_2 + b_2), && B_{32} = a_1, && B_{33} = \tfrac14(a_6 + 5a_5) - a_4,\\
&B_{41} = b_3, && B_{42} = \tfrac14(3b_5 - b_6), && B_{43} = b_7,\\
&B_{51} = \tfrac14(3a_5 - a_6), && B_{52} = a_3, && B_{53} = a_7,\\
&B_{61} = \tfrac14(3b_6 - b_5) - b_4, && B_{62} = \tfrac14(a_5 - 3a_6) + a_4, && B_{63} = a_8 - b_8,
\end{aligned} \tag{3.2}$$

and then the system (1.1) is linearizable to the simplest form $\bar{x}'' = 0$, $\bar{y}'' = 0$;

$$\text{(2) or} \qquad \operatorname{rank} B_1 = 1, \qquad \operatorname{rank} B_2 = 1, \qquad \operatorname{rank} B_3 = 1, \tag{3.3}$$

where B_1 is the first column of matrix B, B_2 is 6×2-dimensional matrix, which consists of the first and the second columns of B, B_3 is 6×2-dimensional matrix, which consists of the first and the third columns of B and following relations hold:

$$B_{j1}(B_{j3,y} - B_{j2,t}) + B_{j2}(B_{j1,t} - B_{j3,x}) + B_{j3}(B_{j2,x} - B_{j1,y}) = 0, \quad j = 1, \ldots, 6, \tag{3.4}$$

$$\begin{aligned}
&B_{j1}^2 B_{j3,t} - B_{j1}B_{j3}(B_{j1,t} + B_{j3,x}) + B_{j3}^2 B_{j1,x} + K_1 B_{j1}^3 - 2L_1 B_{j1}^2 B_{j3}\\
&\qquad + P_1 B_{j1} B_{j3}^2 - V_1 B_{j3}^3 + B_{j2}\big(K_2 B_{j1}^2 - 2M_2 B_{j1} B_{j3} + Q_2 B_{j3}^2\big) = 0,\\
&B_{j1}^2 B_{j3,y} - B_{j1}(B_{j3}B_{j1,y} + B_{j2}B_{j3,x}) + B_{j2}B_{j3}B_{j1,x} + M_1 B_{j1}^3\\
&\qquad - S_1 B_{j1}^2 B_{j3} + V_0 B_{j1} B_{j3}^2 + B_{j2}^2(Q_2 B_{j3} - M_2 B_{j1})\\
&\qquad + B_{j2}\big((L_2 - L_1)B_{j1}^2 + (P_1 - S_2)B_{j1}B_{j3} - V_1 B_{j3}^2\big) = 0,\\
&B_{j1}^2 B_{j2,y} - B_{j1}B_{j2}(B_{j1,y} + B_{j2,x}) + B_{j2}^2 B_{j1,x} + Q_1 B_{j1}^3\\
&\qquad + (P_2 - 2S_1)B_{j1}^2 B_{j2} + (P_1 - 2S_2)B_{j1}B_{j2}^2 + Q_2 B_{j2}^3\\
&\qquad - B_{j3}\big(V_2 B_{j1}^2 - 2V_0 B_{j1}B_{j2} + V_1 B_{j2}^2\big) = 0, \qquad j = 1, \ldots, 6,
\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&B_{j1}[2b_{5t} - b_{4t} + b_{8x} - a_{8x} - b_{7y} + K_1b_1 - K_2B_{31} + 4M_2B_{51} - 2Q_2a_7 \\
&\qquad - (L_1 + L_2)B_{61} + S_2(a_8 - b_8) + (P_2 - 3S_1)b_7] + B_{j3}[b_{6x} - 3b_{5x} + 4V_0b_7] \\
&\qquad + B_{j2}[5b_{7x} - 3b_{3t} + 4M_2B_{42} + Q_2(b_8 - a_8) + (P_1 - 5S_2)b_7] = 0, \\
&B_{j1}[B_{13,t} + B_{33,x} + B_{42,y} - K_1a_0 + Q_1b_3 + 4V_1a_7 + 2V_0(b_8 - a_8) - 2V_2b_7] \\
&\qquad + B_{j2}[-\tfrac{1}{2}b_{1t} - b_{3y} + (P_2 - S_1)b_3 + S_2(B_{61} + B_{42}) + 2V_1(b_8 - a_8) - 2V_0b_7] \\
&\qquad + B_{j3}[2a_{0t} - a_{2x} + \tfrac{1}{2}b_{1y} + 4L_1a_0 - 2M_2b_0 - Q_2a_1 + S_1b_1 - P_1a_2] = 0, \\
&B_{j1}[2a_{4x} - a_{5x} - a_{6x} + 2K_2b_0 - 2Q_2a_3 - 2V_1a_7 + 2V_2b_7] \\
&\qquad + B_{j2}[b_{1t} + 2b_{4x} - b_{5x} - b_{6x} - 2S_1b_3 + 2(P_1 - S_2)B_{42} + 2V_1(a_8 - b_8)] \\
&\qquad + B_{j3}[a_{2x} + b_{2x} - b_{1y} + 2(L_2 - L_1)a_0 + 2Q_2a_1 + 2(S_2 - P_1)B_{13}] = 0, \\
&B_{j1}[b_{2y} - b_{0t} - a_{1x} + (L_2 - 3L_1)b_0 + Q_1b_1 - S_2a_1 + (P_2 - S_1)B_{31} + V_1a_3] \\
&\qquad + B_{j2}[a_{2x} - b_{2x} - 4M_2b_0 - 2V_2b_3 - 2V_0(B_{61} + B_{23}) + V_1(2a_5 - a_4)] \\
&\qquad + B_{j3}[3a_{0y} + 5b_{0x} + (P_1 + 3S_2)b_0 + V_2b_1] = 0, \qquad j = 1, \dots, 6,
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
&B_{j1}\big[a_{0t} + b_{2x} + (L_1 + L_2)a_0 + S_1b_1 + S_2b_2 - \tfrac{1}{2}P_1(a_2 + b_2)\big] \\
&\qquad - B_{j2}[b_{1x} + Q_2B_{22}] + B_{j3}[V_1(a_2 + b_2) - 2V_0b_1] = 0, \\
&B_{j1}[a_{0y} + 3b_{0x} + (P_2 - S_1)a_0 + (3S_2 - P_1)b_0 + V_2b_1] \\
&\qquad + B_{j2}[2a_{0x} - 2Q_2b_0 - 2V_0b_1 + V_1B_{22}] = 0, \\
&B_{j1}[b_{3t} - 3b_{7x} + K_2b_1 - (L_1 + L_2)b_3 + (P_1 + S_2)b_7] \\
&\qquad + B_{j3}[2b_{3x} - 2M_2b_1 + Q_2B_{23} - 2V_1b_7] = 0, \\
&B_{j1}[b_{3y} + b_{4x} - 2b_{5x} + (L_2 - L_1)b_1 + (S_1 - P_2)b_3] + B_{j2}[(3S_2 - P_1)b_3 - 2M_2b_1 \\
&\qquad + Q_2(B_{61} + B_{23})] + B_{j3}[b_{1x} - V_1B_{23} + (P_1 - S_2)b_1] = 0, \\
&B_{j1}\big[\tfrac{1}{2}b_{1t} + b_{4x} - 2b_{5x} - K_2a_0 - S_1b_3 + 2V_0b_7\big] + B_{j3}\big[\tfrac{1}{2}b_{1x} + 2M_2a_0 + Q_2B_{13}\big] \\
&\qquad + B_{j2}[b_{3x} + (P_1 - S_2)b_3 + Q_2B_{42} - 2V_1b_7] = 0, \\
&B_{j1}\big[\tfrac{1}{2}b_{1y} - b_{2x} + (L_1 - L_2)a_0 - 2M_2b_0 - V_2b_3\big] + B_{j2}\big[\tfrac{1}{2}b_{1x} + 2V_0b_3 - V_1B_{42}\big] \\
&\qquad + B_{j3}[-a_{0x} + (S_2 - P_1)a_0 + 2Q_2b_0 - V_1B_{13}] = 0, \qquad j = 1, \dots, 6.
\end{aligned}
\tag{3.7}
$$

Proof. Substituting (1.2) into a linear system (2.2) one obtains a system of second-order ODEs with the same dependence on x', y' as in the system (1.4). Equating corresponding coefficients of x', y' from these two systems to each other leads to 15 relations. When $\theta_x \neq 0$ they can be solved for the following derivatives of φ_1, φ_2, θ:

$$
\begin{aligned}
\varphi_{j,xx} &= V_1\varphi_{j,t} + (2F_0 - P_1)\varphi_{j,x} - Q_2\varphi_{j,y} + 2X_j\theta_x + Z_j\theta_x^2, \\
\varphi_{j,xy} &= V_0\varphi_{j,t} + (F_2 - S_1)\varphi_{j,x} + (F_0 - S_2)\varphi_{j,y} + (X_j\theta_y + Y_j\theta_x) + Z_j\theta_x\theta_y, \\
\varphi_{j,yy} &= V_2\varphi_{j,t} - Q_1\varphi_{j,x} + (2F_2 - P_2)\varphi_{j,y} + 2Y_j\theta_y + Z_j\theta_y^2, \\
\varphi_{j,tx} &= F_0\varphi_{j,t} + (F_1 - L_1)\varphi_{j,x} - M_2\varphi_{j,y} + (T_j\theta_x + X_j\theta_t) + Z_j\theta_t\theta_x, \\
\varphi_{j,ty} &= F_2\varphi_{j,t} - M_1\varphi_{j,x} + (F_1 - L_2)\varphi_{j,y} + (Y_j\theta_t + T_j\theta_y) + Z_j\theta_t\theta_y, \\
\varphi_{j,tt} &= 2F_1\varphi_{j,t} - K_1\varphi_{j,x} - K_2\varphi_{j,y} + 2T_j\theta_t + Z_j\theta_t^2, \qquad j = 1, 2,
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&\theta_{tt} = 2F_1\theta_t - K_1\theta_x - K_2\theta_y, \qquad \theta_{ty} = F_2\theta_t - M_1\theta_x + (F_1 - L_2)\theta_y, \\
&\theta_{yy} = V_2\theta_t - Q_1\theta_x + (2F_2 - P_2)\theta_y.
\end{aligned}
\tag{3.9}
$$

Here we use the notation

$$F_0 = \tfrac{1}{2}\theta_x^{-1}(\theta_{xx} - V_1\theta_t + P_1\theta_x + Q_2\theta_y), \qquad F_1 = \theta_x^{-1}(\theta_{tx} - F_0\theta_t + L_1\theta_x + M_2\theta_y),$$
$$F_2 = \theta_x^{-1}(\theta_{xy} - V_0\theta_t + S_1\theta_x + (S_2 - F_0)\theta_y), \qquad T_j = \tfrac{1}{2}(p_j\varphi_{j,t} + q_j\varphi_{3-j,t}),$$
$$X_j = \tfrac{1}{2}(p_j\varphi_{j,x} + q_j\varphi_{3-j,x}), \qquad Y_j = \tfrac{1}{2}(p_j\varphi_{j,y} + q_j\varphi_{3-j,y}),$$
$$Z_j = l_j\varphi_j + m_j\varphi_{3-j}, \qquad j = 1, 2.$$

Therefore, an invertible change of variables (1.2) with $\theta_x \neq 0$ turns the system of second-order ODEs (1.1) into a linear system (2.2) if and only if the system (1.1) has the form (1.4) and the system of 15 PDEs (3.8) and (3.9) is compatible. Further, we find the compatibility conditions for this system.

Differentiating (3.8) with respect to t, x, y and (3.9) with respect to t, y one can calculate the derivatives $\varphi_{1,txy}$, $\varphi_{2,txy}$ in three different ways and other mixed derivatives in two ways. Comparing them and using expressions (3.8) and (3.9) for the second-order derivatives of φ_1, φ_2, θ gives rise to 18 relations. Their linear combinations take the form of two linear systems for θ_t/θ_x, θ_y/θ_x:

$$B_{j1}\theta_t/\theta_x = B_{j3}, \qquad j = 1, \ldots, 6, \tag{3.10}$$

$$B_{j1}\theta_y/\theta_x = B_{j2}, \qquad j = 1, \ldots, 6, \tag{3.11}$$

three relations for the derivatives θ_{xxx}, θ_{txx}, θ_{xxy}

$$2G_0 = -\alpha_3\theta_x^2 - b_1, \quad G_1 = -\alpha_3\theta_t\theta_x - b_1\theta_t/\theta_x + b_5 + b_6, \quad G_2 = -\alpha_3\theta_x\theta_y - b_2, \tag{3.12}$$

and three equations

$$\begin{aligned} \alpha_0 &= \Delta^{-1}\theta_x^{-3}(b_3M_{21}M_{31} + b_1(M_{21}M_{33} + M_{23}M_{31})/2 - a_0M_{23}M_{33}),\\ \alpha_1 &= \Delta^{-1}\theta_x^{-3}\big(a_0M_{33}^2 - b_1M_{31}M_{33} - b_3M_{31}^2\big),\\ \alpha_2 &= \Delta^{-1}\theta_x^{-3}\big(b_3M_{21}^2 + b_1M_{21}M_{23} - a_0M_{23}^2\big), \end{aligned} \tag{3.13}$$

where coefficients B_{ij} are given by (3.2), M_{ij} denotes the minor of the corresponding element in Jacobi matrix of transformation (1.2), $\alpha_i(\theta)$, $i = 0, 1, 2, 3$ are given by (2.9) and

$$\begin{aligned} G_0 &= F_{0x} - F_0^2 + P_1F_0 + Q_2F_2 - V_{1t} - (L_1 + F_1)V_1 - M_2V_0,\\ G_1 &= F_{1x} + (L_1 - F_1)F_0 + M_2F_2 + b_6 - \tfrac{1}{2}K_1V_1 - \tfrac{1}{2}K_2V_0 + G_0\theta_t/\theta_x + b_3\theta_y/\theta_x,\\ G_2 &= F_{2x} + (S_1 - F_2)F_0 + S_2F_2 - V_{0t} - M_1V_1 - (L_2 + F_1)V_0 + a_0\theta_t/\theta_x + G_0\theta_y/\theta_x. \end{aligned}$$

According to the Kronecker–Capelli theorem the system (3.10) is consistent if and only if $\operatorname{rank} B_1 = \operatorname{rank} B_3$ while the system (3.11) is consistent if and only if $\operatorname{rank} B_1 = \operatorname{rank} B_2$. It is possible when either $\operatorname{rank} B = 0$ or relations (3.3) hold.

First consider the case of $\operatorname{rank} B_i = 1$, $i = 1, 2, 3$. Compatibility conditions of systems (3.10) and (3.11) arise as difference of equations (3.11) differentiated with respect to t and the corresponding equations (3.10) differentiated with respect to y. These conditions take the form (3.4) when θ_{tx}, θ_{xy} are eliminated using relations (3.10) and (3.11) differentiated with respect to x. Equalities (3.5) arise in a similar manner as compatibility conditions of equations (3.10) and (3.11) with equations (3.9).

Equalities (3.6) are obtained as compatibility conditions of equations (3.9) and (3.12) in the following way. The second and third equations (3.12) differentiated with respect to t, y are solved for the derivatives θ_{ttxx}, θ_{txxy}, θ_{xxyy} and compared with the same derivatives arising from expressions (3.9) differentiated twice with respect to x. After replacing the derivatives of θ by virtue of (3.9)–(3.12) these relations take the form (3.6).

Differentiation of relations (3.13) with respect to t, x, y, elimination of α_0', α_1', α_2' and substitution of second-order derivatives of the functions φ_1, φ_2, θ by virtue of (3.8)–(3.11) lead to equalities

$$\begin{aligned}
&a_{0t} + b_{2x} - b_{1x}\theta_y/\theta_x + a_0H_1 - b_1H_3 = 0,\\
&a_{0y} + 3b_{0x} + 2a_{0x}\theta_y/\theta_x + a_0H_2 - b_1H_4 = 0,\\
&b_{3t} - 3b_{7x} + 2b_{3x}\theta_t/\theta_x + b_1H_5 - b_3H_1 = 0,\\
&b_{3y} + b_{4x} - 2b_{5x} + b_{1x}\theta_t/\theta_x + b_1H_6 - b_3H_2 = 0,\\
&\tfrac{1}{2}b_{1t} + b_{4x} - 2b_{5x} + \tfrac{1}{2}b_{1x}\theta_t/\theta_x + b_{3x}\theta_y/\theta_x + b_3H_3 - a_0H_5 = 0,\\
&\tfrac{1}{2}b_{1y} - b_{2x} + \tfrac{1}{2}b_{1x}\theta_y/\theta_x - a_{0x}\theta_t/\theta_x + b_3H_4 - a_0H_6 = 0,
\end{aligned} \tag{3.14}$$

with the coefficients

$$\begin{aligned}
&H_1 = L_1 + L_2 - (P_1 + S_2)\theta_t/\theta_x + 2V_1\theta_t^2/\theta_x^2,\\
&H_2 = P_2 - S_1 + (P_1 - 3S_2)\theta_y/\theta_x + 2Q_2\theta_y^2/\theta_x^2,\\
&H_3 = -S_1 + (P_1 - S_2)\theta_y/\theta_x + Q_2\theta_y^2/\theta_x^2 + 2V_0\theta_t/\theta_x - 2V_1\theta_t\theta_y/\theta_x^2,\\
&H_4 = -V_2 + 2V_0\theta_y/\theta_x - V_1\theta_y^2/\theta_x^2,\\
&H_5 = K_2 - 2M_2\theta_t/\theta_x + Q_2\theta_t^2/\theta_x^2,\\
&H_6 = L_2 - L_1 + (P_1 - S_2)\theta_t/\theta_x - V_1\theta_t^2/\theta_x^2 - 2M_2\theta_y/\theta_x + 2Q_2\theta_t\theta_y/\theta_x^2.
\end{aligned}$$

Using (3.10) and (3.11), equalities (3.14) multiplied by B_{j1} are changed to relations linear in θ_t/θ_x, θ_y/θ_x and then to the form (3.7).

It remains to find compatibility conditions for equations (3.12). One should differentiate the second and third equations (3.12) with respect to x, the first one with respect to t, y and compare the derivatives θ_{txxx} and θ_{xxxt}, θ_{yxxx} and θ_{xxxy}. This yields the equalities which hold identically, taking into account (3.9)–(3.11) and (3.14).

For the case of $\operatorname{rank} B = 0$ equations (3.10) and (3.11), compatibility conditions for equations (3.12) as well as for equations (3.9) with (3.12) become identities. The right-hand side of relations (3.13) vanishes. Thus, the system (1.1) is reducible to a linear system, which satisfies the conditions $\alpha_0 = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$. From the results of [9] (corollary 1 of theorem 2) it follows that in this case the system (1.1) is linearizable to the form $\bar{x}'' = 0$, $\bar{y}'' = 0$. □

Remark 1. To prove linearizability of the system (1.1) via a point transformation (1.2) with $\theta_y \neq 0$ it will suffice to make the substitution $\bar{x} = y$, $\bar{y} = x$ into this system and then apply theorem 3.1

Remark 2. Conditions (3.4)–(3.7) are rather cumbersome, but they can be programmed in any symbolic package such as Maple or Mathematica. Moreover, to establish nonlinearizability of systems (1.1), for the most part it is sufficient to check conditions (3.3) only. Examples 3 and 5 illustrate this in the next section.

For the case of linearizability by transformation (1.2) with $\theta_y \neq 0$ condition analogous to condition (3.3) looks like

$$\operatorname{rank} B_2 = 1, \qquad \operatorname{rank} B_4 = 1, \qquad \operatorname{rank} B_5 = 1, \tag{3.15}$$

where B_4 is the second column of matrix B, B_5 is 6×2-dimensional matrix, which consists of the second and the third columns of B. Condition (3.15) for the system (1.1) becomes condition (3.3) for this system rewritten in variables $\bar{x} = y$, $\bar{y} = x$.

If given the system (1.1) is linearizable, then there are two ways for finding the corresponding transformation (1.2) (or (2.1)). The first is to integrate the compatible system of PDEs (3.8)–(3.13) (or (2.6), (2.8)). The second is to calculate Lie symmetry algebra of the system (1.1), find a four-dimensional subalgebra satisfying theorems 1.2 or 1.3 and then construct the change of variables which maps this subalgebra to one of realizations given in [10, sections 3–4].

4. Applications

Now consider a few examples to illustrate the theorems.

Example 1. In [12] while studying the movement of zeros for a linear PDE, a system of n second-order ODEs has been obtained. It may be treated as a system which represents the motion of n interacting particles. For $n = 2$ it takes the form of equations

$$\begin{aligned} x'' &= B_0 + B_1 x - Ex' + (x-y)^{-1}[2(A_0 + A_1 x + A_2 x^2 + A_3 x^2 y) + 2x'y' \\ &\qquad - (D_0 + D_1 x)(x'+y') - D_2 x(xy' + yx')], \\ y'' &= B_0 + B_1 y - Ey' + (y-x)^{-1}[2(A_0 + A_1 y + A_2 y^2 + A_3 x y^2) + 2x'y' \\ &\qquad - (D_0 + D_1 y)(x'+y') - D_2 y(xy' + yx')], \end{aligned} \tag{4.1}$$

with constant parameters A_0, A_1, A_2, A_3, B_0, B_1, D_0, D_1, D_2, E. For the system (4.1) only four coefficients (1.6) are nonzero:

$$a_7 = \frac{\lambda_0 + \lambda_1 x + \lambda_2 x^2}{4(y-x)}, \qquad a_8 = \lambda_3 + \frac{\lambda_0 + \lambda_1 x + \lambda_2 xy}{4(y-x)},$$
$$b_7 = \frac{\lambda_0 + \lambda_1 y + \lambda_2 y^2}{4(x-y)}, \qquad b_8 = \lambda_3 + \frac{\lambda_0 + \lambda_1 y + \lambda_2 xy}{4(x-y)},$$

where $\lambda_0 = D_0(D_1 + 2E) - 4B_0$, $\lambda_1 = D_1(D_1 + 2E) - 4B_1$, $\lambda_2 = D_2(D_1 + 2E) + 8A_3$, $\lambda_3 = (D_0 D_2 - E^2)/4 - 2(A_2 + B_1)$. It is readily verified that the system (4.1) satisfies all conditions (2.4) and (2.5) of theorem 2.1. In variables $\bar{x} = x + y$, $\bar{y} = xy$ it becomes

$$\bar{x}'' = -(D_1 + E)\bar{x}' - D_2\bar{y}' + (2A_2 + B_1)\bar{x} + 2A_3\bar{y} + 2A_1,$$
$$\bar{y}'' = D_0\bar{x}' - E\bar{y}' + B_0\bar{x} + 2(A_2 + B_1)\bar{y} - 2A_0.$$

Moreover, if λ_0, λ_1, λ_2 are equal to zero, then the system (4.1) satisfies conditions (1.5) of theorem 1.1. In this case let us find transformation (2.1) linearizing (4.1) to the form $\bar{x}'' = 0$, $\bar{y}'' = 0$. Coefficients (2.9) equal to zero and the system (2.8) reduce to (2.10):

$$2\theta'\theta''' - 3\theta''^2 = 4\lambda_3\theta'^2. \tag{4.2}$$

Equation (4.2) has a particular solution $\theta = \tan\sqrt{\lambda_3}t$, $\theta = \tanh\sqrt{-\lambda_3}t$ or $\theta = t$ for λ_3 positive, negative or zero, respectively. The last three pairs of equations (2.6)

$$\varphi_{j,xx} = 0, \qquad \varphi_{j,xy} = (\varphi_{j,y} - \varphi_{j,x})/(x-y), \qquad \varphi_{j,yy} = 0, \qquad j = 1, 2,$$

have the solution $\varphi_j = \alpha_j(t)xy + \beta_j(t)(x+y) + \gamma_j(t)$, $j = 1, 2$. The remaining equations (2.6) take the form

$$\begin{aligned} &2\alpha' = (\theta''/\theta' + E)\alpha + D_2\beta, \qquad 2\beta' = (\theta''/\theta' + D_1 + E)\beta - D_0\alpha, \\ &\gamma'' = \gamma'\theta''/\theta' + 2A_0\alpha - 2A_1\beta, \end{aligned} \tag{4.3}$$

where we omit the lower indices for α, β, γ. For a given $\theta = \theta(t)$ let $(\alpha_j(t), \beta_j(t))$, $j = 1, 2$ be the fundamental system of solutions of the first two equations (4.3). Then it is not difficult to find corresponding particular solutions $\gamma_1(t)$, $\gamma_2(t)$ of the last equation (4.3) and thus obtain the linearizing transformation (2.1).

Example 2. In [13] all systems of two second-order ODEs admitting real four-dimensional Lie algebras are classified. Consider one of them, which is not decoupled and has the form (1.4) (cf system $A_{17}^{(3)}$ in [6])

$$x'' + f(y)(x'^2 + 1)y' = 0, \qquad y'' + (f(y)x' + g(y))y'^2 = 0. \tag{4.4}$$

For these equations all coefficients (1.6) vanish except for $a_3 = -b_0 = (f' - fg)/2$. Matrix B has two nonzero elements (3.2) $B_{21} = b_0$ and $B_{25} = a_3$. If we have $f(y) = 2\exp F(y)$, $g(y) = F'(y)$ with some function $F(y)$, then $a_3 = 0$, $b_0 = 0$ and the system (4.4) satisfy condition (3.1) of theorem 3.1. Therefore, it is linearizable to the form $\bar{x}'' = 0$, $\bar{y}'' = 0$ by a point transformation (1.2) with $\theta_x \neq 0$.

If $f' - fg \neq 0$, the system (4.4) does not satisfy condition (3.3) of theorem 3.1, but it satisfies condition (3.15). Interchanging x and y one obtains the system (with $b_3 = -a_0 \neq 0$), which satisfies all conditions (3.3)–(3.7) of theorem 3.1. It means that the system (4.4) is linearizable by a point transformation (1.2) with $\theta_y \neq 0$, $\theta_x = 0$. It is not difficult to see that simple change of variables

$$\bar{t} = y, \qquad \bar{x} = x, \qquad \bar{y} = t$$

turns the system (4.4) into a linear system

$$\bar{x}'' = g(\bar{t})\bar{x}' - f(\bar{t})\bar{y}', \qquad \bar{y}'' = g(\bar{t})\bar{y}' + f(\bar{t})\bar{x}'.$$

Now let us find transformation (1.2) linearizing the system

$$x'' + 2\mathrm{e}^{F(y)}(x'^2 + 1)y' = 0, \qquad y'' + (2x'\mathrm{e}^{F(y)} + F'(y))y' = 0 \tag{4.5}$$

to the form $\bar{x}'' = 0$, $\bar{y}'' = 0$. The first equation (3.12), $2\theta_{xxx}/\theta_x - 3\theta_{xx}^2/\theta_x^2 = 0$, is satisfied, for example, by linear function in x. Substitution of $\theta = x + t\Omega(y)$ into equations (3.9) and remaining equations (3.12) leads to a single relation $\Omega' = (1 + \Omega^2)\exp F(y)$. Equations (3.8) become

$$\begin{array}{lll} \varphi_{j,xx} = 0, & \varphi_{j,xy} = \mathrm{e}^F(\Omega\varphi_{j,x} - \varphi_{j,t}), & \varphi_{j,yy} = (F' + 2\Omega\mathrm{e}^F)\varphi_{j,y}, \\ \varphi_{j,tx} = 0, & \varphi_{j,ty} = \mathrm{e}^F(\Omega\varphi_{j,t} + \varphi_{j,x}), & \varphi_{j,tt} = 0, \qquad j = 1, 2. \end{array}$$

Their particular solution is given by $\varphi_1 = t - x\Omega(y)$, $\varphi_2 = \Omega(y)$. Thus, the system (4.5) is mapped to equations (1.3) by the transformation

$$\bar{t} = x + t\Omega(y), \qquad \bar{x} = t - x\Omega(y), \qquad \bar{y} = \Omega(y), \qquad \Omega(y) = \tan\left(\int \mathrm{e}^{F(y)}\,\mathrm{d}y\right).$$

Example 3. As is shown in [14], the Heisenberg model $\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}$, $S_1^2 + S_2^2 + S_3^2 = 1$ via the transformation

$$S_1 = \frac{2}{u + v}, \qquad S_2 = \frac{2uv}{u + v}, \qquad S_3 = \frac{u - v}{u + v}$$

reduces to the system

$$u_t = u_{xx} - \frac{2u_x^2}{u + v}, \qquad -v_t = v_{xx} - \frac{2v_x^2}{u + v}.$$

The system of two second-order ODEs

$$u'' = \frac{2u'^2}{u + v}, \qquad v'' = \frac{2v'^2}{u + v} \tag{4.6}$$

describes its stationary solution. Equations (4.6) admit five-dimensional Lie algebra L_5 with the basis [15]

$$X_1 = \partial_x, \quad X_2 = x\partial_x, \quad X_3 = \partial_u - \partial_v, \quad X_4 = u\partial_u + v\partial_v, \quad X_5 = u^2\partial_u - v^2\partial_v.$$

The nonzero commutators are

$$[X_1, X_2] = X_1, \qquad [X_3, X_4] = X_3, \qquad [X_3, X_5] = 2X_4, \qquad [X_4, X_5] = X_5.$$

To establish nonlinearizability of the system (4.6) we can use theorems 1.2 or 1.3. But in this case we need to prove that algebra L_5 does not contain subalgebra $L_{4.1}$ or $L_{4.2}$ (up to transformations of the basis of Lie algebra).

Here it is easier to use theorems 2.1 and 3.1. For the system (4.6) the only nonzero coefficients (1.6) are $a_2 = b_2 = 2(u+v)^{-2}$ and condition (2.4) of theorem 2.1 is not satisfied. The nonzero elements (3.2) of matrix B are $B_{31} = B_{22} = 2(u+v)^{-2}$ whence it follows that $\operatorname{rank} B_2 = 2$. System (4.6) does not satisfy condition (3.3) of theorem 3.1 or condition (3.15) either. Hence, it is not linearizable.

Example 4. System (4.6) belongs to the class of equations

$$\begin{aligned} x'' &= P_1(x,y)x'^2 + 2S_1(x,y)x'y' + Q_1(x,y)y'^2, \\ y'' &= P_2(x,y)y'^2 + 2S_2(x,y)x'y' + Q_2(x,y)x'^2, \end{aligned} \tag{4.7}$$

which one can associate with geodesic equations

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0, \qquad i, j, k, = 1, 2. \tag{4.8}$$

If we set $(x^1, x^2) = (x, y)$, then the coefficients of projective connection Γ^i_{jk} are related to coefficients of the system (4.7) as follows:

$$\Gamma^1_{11} = -P_1, \quad \Gamma^1_{12} = -S_1, \quad \Gamma^1_{22} = -Q_1, \quad \Gamma^2_{11} = -Q_2, \quad \Gamma^2_{12} = -S_2, \quad \Gamma^2_{22} = -P_2. \tag{4.9}$$

For the system (4.7) nonzero coefficients (1.6) are

$$\begin{aligned} a_1 &= Q_{1x} - S_{1y} + Q_1(S_2 - P_1) + S_1(S_1 - P_2), \quad & a_2 &= S_{1x} - P_{1y} + Q_1Q_2 - S_1S_2, \\ b_1 &= Q_{2y} - S_{2x} + Q_2(S_1 - P_2) + S_2(S_2 - P_1), \quad & b_2 &= S_{2y} - P_{2x} + Q_1Q_2 - S_1S_2. \end{aligned} \tag{4.10}$$

If a_1, a_2, b_1, b_2 vanish, then the system (4.7) satisfies theorems 2.1 and 1.1. It is mapped to the form $\bar{x}'' = 0$, $\bar{y}'' = 0$ by the transformation $\bar{t} = t$, $\bar{x} = \varphi_1(x,y)$, $\bar{y} = \varphi_2(x,y)$, where φ_1, φ_2 are functionally independent solutions of the compatible system of equations (cf [16, theorem 1]):

$$\varphi_{xx} + P_1\varphi_x + Q_2\varphi_y = 0, \quad \varphi_{xy} + S_1\varphi_x + S_2\varphi_y = 0, \quad \varphi_{yy} + Q_1\varphi_x + P_2\varphi_y = 0. \tag{4.11}$$

The explicit procedure for constructing this linearizing transformation is given in [7, 16].

For the system (4.7) matrix B is given by

$$B^T = \begin{pmatrix} 0 & b_1 & (3a_2+b_2)/4 & 0 & 0 & 0 \\ 0 & (a_2+3b_2)/4 & a_1 & 0 & 0 & 0 \\ (b_2-a_2)/4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and condition (3.3) of theorem 3.1 is satisfied if and only if the relations

$$b_2 = a_2, \qquad a_1b_1 - a_2b_2 = 0, \qquad b_1 \neq 0 \tag{4.12}$$

hold. In this case equalities (3.4) hold identically, (3.6) and (3.7) provide the relations

$$\begin{aligned} &b_{1y} - a_{2x} + S_1 b_1 + (S_2 - P_1)a_2 - Q_2 a_1 = 0, \qquad b_1 a_{1x} - a_2 b_{1y} = 0, \\ &a_{2y} - a_{1x} + Q_1 b_1 + (P_2 - S_1)a_2 - S_2 a_1 = 0, \qquad a_2 a_{1x} - a_1 b_{1y} = 0 \end{aligned} \tag{4.13}$$

and (3.5) become identities by virtue of (4.12) and (4.13).

In order to find linearizing transformation (1.2), we note that for the system (4.7) relations

$$\begin{aligned} &F_0 = (\theta_{xx} + 2P_1\theta_x + Q_2\theta_y)/(2\theta_x), \qquad F_1 = 0, \qquad F_2 = F_0\theta_y/\theta_x, \\ &G_0 = F_{0x} - F_0^2 + P_1 F_0 + Q_2 F_2, \qquad G_1 = 0, \qquad G_2 = 2G_0\theta_y/\theta_x \end{aligned}$$

hold and the system (3.9)–(3.12) reduces to simultaneous equations

$$\begin{aligned} \theta_t = 0, \qquad b_1\theta_y - a_2\theta_x = 0, \qquad &2\theta_{xxx}/\theta_x - 3\theta_{xx}^2/\theta_x^2 + 2\alpha_3\theta_x^2 + 2b_1 + 2P_{1x} \\ &+ P_1^2 - 2Q_2S_1 + 2(Q_{2x} + Q_2(2P_1 - S_2))a_2/b_1 + 3Q_2^2 a_1/b_1 = 0 \end{aligned} \tag{4.14}$$

in the function θ. From the last equations (3.8), $\varphi_{j,tt} = 0$, $j = 1, 2$, we have

$$\varphi_j = t\beta_j(x, y) + \gamma_j(x, y), \qquad j = 1, 2.$$

Then equations (3.8) with $\varphi_{j,tx}$, $\varphi_{j,ty}$ reduce to

$$\beta_{j,x} = F_0\beta_j + (p_j\beta_j + q_j\beta_{3-j})\theta_x/2, \qquad \beta_{j,y} = \beta_{j,x}\theta_y/\theta_x, \qquad j = 1, 2.$$

We can take the linearizing transformation in the form

$$\bar{t} = \theta(\tau), \qquad \bar{x} = t\beta(\tau), \qquad \bar{y} = \gamma(x, y), \tag{4.15}$$

where $\tau(x, y) = \text{const}$ provides the general solution of ODE $\mathrm{d}x/\mathrm{d}y + a_2/b_1 = 0$, $\theta_x\gamma_y - \theta_y\gamma_x \neq 0$ and functions θ, β, γ are the solutions of equations (4.14) and

$$\begin{aligned} &\beta_x = (F_0 + p_1\theta_x/2)\beta, \qquad \gamma_{xx} = (2F_0 - P_1)\gamma_x - Q_2\gamma_y + p_2\theta_x\gamma_x + l_2\theta_x^2\gamma, \\ &\gamma_{xy} = (F_2 - S_1)\gamma_x + (F_0 - S_2)\gamma_y + p_2(\theta_x\gamma_y + \theta_y\gamma_x)/2 + l_2\theta_x\theta_y\gamma, \\ &\gamma_{yy} = -Q_1\gamma_x + (2F_2 - P_2)\gamma_y + p_2\theta_y\gamma_y + l_2\theta_y^2\gamma. \end{aligned}$$

System (4.7) satisfying conditions (4.12) and (4.13) is mapped by transformation (4.15) to a decoupled linear system.

Remark. While the system (4.7) satisfying (4.12) and (4.13) is not reducible to the simplest equations $\bar{x}'' = 0$, $\bar{y}'' = 0$, in this case in the system (4.7) an equation $\bar{x}'' = 0$ separates in variables $\bar{t} = t$, $\bar{x} = \varphi(x, y)$. Function $\varphi(x, y)$ should satisfy equations (4.11), whose compatibility conditions $b_1\varphi_y - a_2\varphi_x = 0$, $b_2\varphi_y - a_1\varphi_x = 0$ coincide by virtue of (4.12).

Example 5. Consider two-dimensional space with the metric $g(x^1, x^2)$ defined by symmetric nondegenerate matrix g_{ij}. Then the coefficients of geodesic equations (4.8) are the Christoffel symbols given in terms of the metric tensor g_{ij} as

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}), \tag{4.16}$$

where the '$_{,j}$' stands for the partial derivative with respect to x^j and g^{ij} is the inverse matrix for g_{ij}. The Riemann curvature tensor is defined by

$$R^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk}. \tag{4.17}$$

Substituting (4.16) into (4.17) and comparing with (4.9) and (4.10) it is readily seen that for a system of two geodesic equations we have

$$\begin{aligned} &a_1 = R^1_{221} = \frac{g_{22}A}{4\Delta^2}, \quad a_2 = R^1_{121} = \frac{g_{12}A}{4\Delta^2}, \quad b_2 = R^2_{212} = \frac{g_{12}A}{4\Delta^2}, \quad b_1 = R^2_{112} = \frac{g_{11}A}{4\Delta^2}, \\ &\Delta = g_{11}g_{22} - g_{12}^2, \quad A = 2\Delta(g_{11,22} + g_{22,11} - 2g_{12,12}) \\ &\qquad + g_{11}\big(g_{22,2}(2g_{12,1} - g_{11,2}) - g_{22,1}^2\big) + g_{22}\big(g_{11,1}(2g_{12,2} - g_{22,1}) - g_{11,2}^2\big) \\ &\qquad + g_{12}(g_{11,2}g_{22,1} - g_{11,1}g_{22,2} + 2g_{12,1}g_{22,1} + 2g_{12,2}g_{11,2} - 4g_{12,1}g_{12,2}). \end{aligned} \tag{4.18}$$

The following question naturally arises: is it possible to reduce the system of geodesic equations (4.8) to a linear system that is not equivalent to the simplest equations

$$\frac{\mathrm{d}^2\bar{x}^i}{\mathrm{d}\bar{t}^2} = 0, \qquad i = 1, 2. \tag{4.19}$$

To find the answer we should check conditions (4.12) and (4.13) for the system (4.8). The first condition (4.12), $b_2 = a_2$, is obviously satisfied. Substitution of (4.18) into the second one, $a_1b_1 - a_2b_2 = 0$, yields the relation $A/4\Delta = 0$, whence it follows that a_1, a_2, b_1, b_2 vanish. But in this case the system (4.8) satisfies theorem 1.1 and, therefore, is reducible to equations (4.19). Thus, in metric space the only possibility for a system of two geodesic equations is the linearization to the form (4.19) provided the Riemann tensor is zero. This fact that the space is flat if and only if the Riemann tensor vanishes is well known (see, e.g., [6, 7, 16]).

5. Conclusion

In this paper we study nonlinear systems reducible to arbitrary linear systems, which may not be equivalent to the restricted class of equations $x'' = 0$. The importance of this problem has been noticed, for example, in [7]. Here we prove general linearizability criteria for a system of two second-order ODEs, which are given in terms of coefficients of the system. A similar problem for a system of three second-order ODEs remains open.

One of the possible applications of our results may be a further extension of the method of finding sequentially linearizable systems proposed in [11]. Applying this method to the systems of three second-order ODEs, first the number of dependent variables of the system should be reduced, then the linearization criteria (theorems 1.1, 2.1 or 3.1) may be used for the reduced system of two equations.

In the paper it is shown that the class of systems of two second-order quadratically semi-linear ODEs involves the systems which, being linearizable, are not reducible to $x'' = 0$. However, it is established that such systems cannot be treated as systems of two geodesic equations in metric space. It is shown that systems of two geodesic equations may be linearizable only to the form $x'' = 0$. The study of the linearization problem (as in section 4) for systems of n geodesic equations remains an open problem.

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