Functional Analysis and Its Applications, Vol. 43, *No.* 4, *pp.* ??–??, 2009
Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 43, *No.* 4, *pp.* 87–90, 2009

BRIEF COMMUNICATIONS

Three Series of Invariant Manifolds of the Sawada–Kotera Equation*

Yu. Yu. Bagderina

Received March 14, 2008

Abstract. We find a new infinite sequence of invariant manifolds for the Sawada–Kotera equation, in addition to the known two sequences of its symmetries and conservation laws. The elements of these three sequences are related cyclically by recursion relations similar to the Lenard formula for the KdV equation. For any $n > 0$, there are two invariant manifolds of order $2n$, which allows one to construct two n-soliton solutions of the Sawada–Kotera equation.

Key words: evolution equation, invariant manifold, symmetry, conservation law, soliton solution.

1. Introduction. The Sawada–Kotera equation ([1], [2])

$$u_t = F[u] \equiv u_5 - 30uu_3 - 30u_1u_2 + 180u^2u_1, \qquad u_j = \partial^j u/\partial x^j, \tag{1}$$

as well as the Korteweg–de Vries (KdV) equation ([3], [4]) admits the Hamiltonian representation $u_t = P^J \frac{\delta H}{\delta u}$ with Poisson operator $P^J = D^3 - 12(uD+Du)$ and Hamiltonian $H = \int(-u_1^2/2 - u^3)\,dx$. It was established in [5] that Eq. (1) is integrable by the inverse scattering method and possesses an infinite series of polynomial conservation laws $D_tT_i = DS_i$, where D_t and D stand for the total derivatives with respect to t and x, respectively. A factorization of the Lax operator of Eq. (1) was used in [6] and [7] to find a Miura transformation relating Eq. (1) with the modified equation. The bi-Hamiltonian structure ([8], [9]) of Eq. (1) permitted Fuchssteiner and Oevel [10] to obtain a countable set of its symmetries $X = f_i\,\partial_u$ and conservation laws. The functions $f_i = DJ_i$ and the variational derivatives L_i of the conserved densities are related by the recursion formulas

$$DJ_{i+1} = P^J L_i, \qquad L_{i+2} = (D^3 - 6(D^2uD^{-1} + D^{-1}uD^2) + 18(u^2D^{-1} + D^{-1}u^2))DJ_i. \tag{2}$$

Since formulas (2) are too cumbersome, it was not proved in [10] that the conservation laws are in involution. The equations $J_i = 0$, $L_i = 0$, $DJ_i = 0$, and $DL_i = 0$ define invariant manifolds ([11], [12]) of Eq. (1).

Definition. One says that an ordinary differential equation (ODE)

$$f(x, u, u_1, \dots, u_k) = 0 \tag{3}$$

defines an *invariant manifold* of an evolution equation $u_t = F[u] \equiv F(x, u, u_1, \dots, u_m)$ if

$$X(F)f\big|_{[f]} = 0, \qquad X(F) = F\,\partial_u + \sum_{j\geqslant 1} D^jF\,\partial_{u_j}, \tag{4}$$

where $[f]$ is the manifold defined by Eq. (3) and its differential corollaries with respect to x, namely, the equations $D^jf = 0$, $j = 1, \dots, m$.

Note that it follows from the criterion

$$X(f)(u_t - F)\big|_{u_t=F} = X(F)f - X(f)F = 0 \tag{5}$$

*Supported by grants nos. MK-5863.2008.1, SS-2215.2008.1, and RFBR 09-01-92431-KE.

of invariance of the equation $u_t = F[u]$ under the symmetry operator

$$X(f) = f\,\partial_u + \sum_{j=1}^{m} D^j f\,\partial_{u_j} + D_t f\,\partial_{u_t}$$

that (4) holds, but the converse is not true. The same pertains to the conservation laws ([12], [13]).

In what follows, we show that, in contrast to the case of the KdV equation [11], the manifolds corresponding to symmetries and conservation laws are not the only invariant manifolds of Eq. (1).

2. Invariant manifolds of the Sawada–Kotera equation. In addition to known two series of symmetries and conservation laws [10], we have found a third infinite sequence of invariant manifolds of equation (1). The terms of these three sequences can be calculated recursively: the symmetries $X = DJ_i\,\partial_u$ are defined by the relations

$$DJ_{i+1} = P^J L_i, \quad P^J = D^3 - 24uD - 12u_1, \quad J_{-1} = -1/6; \tag{6}$$

the left-hand sides of the equations $U_i = 0$ that define the earlier unknown invariant manifolds are found from the relations

$$DU_{i+1} = P^U J_i, \quad P^U = D^3 - 6uD - 6u_1, \tag{7}$$

or, integrating once, $U_{i+1} = D^2 J_i - 6uJ_i$; and the conserved densities are determined from the relations

$$DL_{i+1} = P^L U_i, \quad P^L = D^3 - 6uD, \quad L_{-1} = -1/12. \tag{8}$$

It is not difficult to compute the first few terms of these sequences (the constants of integration are taken equal to zero):

$$\begin{gathered} U_0 = u, \quad U_1 = u_2 - 6u^2, \\ U_3 = u_6 - 6(6uu_4 + 10u_1u_3 + 5u_2^2) + 360(u^2u_2 + uu_1^2 - u^4), \quad \dots, \\ J_0 = u, \quad J_2 = u_4 - 30uu_2 + 60u^3, \\ J_3 = u_6 - 42(uu_4 + u_1u_3 + u_2^2) + 252(2u^2u_2 + uu_1^2 - 2u^4), \quad \dots, \\ L_1 = u_2 - 3u^2, \quad L_2 = u_4 - 18uu_2 - 9u_1^2 + 24u^3, \quad \dots . \end{gathered} \tag{9}$$

Every third term in the sequences J_i, U_i, and L_i is missing:

$$L_{3k} = 0, \quad J_{3k+1} = 0, \quad U_{3k+2} = 0, \qquad k = 0, 1, 2, \dots . \tag{10}$$

Formulas (6), (8), and (7) applied consecutively lead to the relation

$$J_{i+3} = D^{-1}P^J D^{-1} P^L D^{-1} P^U J_i$$

for J_i. In view of the factorizations $P^L = (D^2 - 6u)D$ and $P^U = D(D^2 - 6u)$, this implies the well-known formula

$$f_{i+3} = Rf_i, \qquad R = (D^2 - 24u - 12u_1D^{-1})(D^2 - 6u)D(D^2 - 6u)D^{-1},$$

for the symmetries $f_i = DJ_i$ of Eq. (1) with the recursion operator R discovered by V. V. Sokolov and A. B. Shabat. Thus, the operator P^U in (7), which defines an additional series of invariant manifolds of Eq. (1), is one of the factors of the operator R.

If the constants K_i arising in integrating relations (6)–(8) are not set to zero, then the following statement holds.

Theorem. *Equation* (1) *has countably many invariant manifolds defined by the equations*

$$\begin{gathered} R^J_{2n}[u] \equiv J_n + \sum_{i=0}^{n-1} K_i J_i + K_{-1} = 0, \quad R^U_{2n}[u] \equiv U_n + \sum_{i=0}^{n-1} K_i U_i + K_{-1} = 0, \\ R^L_{2n}[u] \equiv L_n + \sum_{i=0}^{n-1} K_i L_i + K_{-1} = 0, \qquad K_{-1}, \dots, K_{n-1} = \text{const}, \end{gathered} \tag{11}$$

where J_i, U_i, and L_i belong to the algebra A of polynomials in the function u and its derivatives with respect to x and satisfy the recursion formulas (6)–(8). Here the $X(DJ_i) = DJ_i\,\partial_u$ are the symmetries, and the L_i are the variational derivatives of conserved densities of Eq. (1); the functionals T_i corresponding to the derivatives L_i are in involution with respect to the Poisson bracket

$$\{T_i, T_j\} \equiv \int \frac{\delta T_i}{\delta u} P^J \frac{\delta T_j}{\delta u}\,dx = \int L_i P^J L_j\,dx = \int L_i DJ_{j+1}\,dx = (L_i, DJ_{j+1}).$$

Proof. The assertion concerning the symmetries $X(DJ_i) = DJ_i\,\partial_u$ was proved in [10]. By eliminating U_i from (7) and (8), we arrive at the relation $DL_{i+2} = (D^5 - 6(2uD^3 + 3u_1D^2 + 3u_2D + u_3) + 36(u^2D + uu_1))J_i$, which coincides with the result obtained by applying the operator D to the second relation in (2).

Let us show that the ODE $U_{i+1} = 0$ defines an invariant manifold of Eq. (1), which acquires the form $u_t = DJ_2$ in notation (9). Using relation (5) in the form $X(DJ_2)DJ_i = X(DJ_i)DJ_2$ and rearranging terms in the chain of equations

$$\begin{aligned} X(DJ_2)U_{i+1} &= (D^2 - 6u)X(DJ_2)J_i - 6J_iDJ_2 \\ &= (D^2 - 6u)D^{-1}(X(DJ_2)DJ_i) - 6J_iDJ_2 \\ &= (D^2 - 6u)X(DJ_i)J_2 - 6J_iDJ_2 \\ &= (D^2 - 6u)(D^5J_i - 30uD^3J_i + (180u^2 - 30u_2)DJ_i) - 6J_iD(u_4 - 30uu_2 + 60u^3) \\ &= (D^5 - 30uD^3 - 30u_1D^2 + 180u^2D)(D^2 - 6u)J_i \\ &= D^5U_{i+1} - 30uD^3U_{i+1} - 30u_1D^2U_{i+1} + 180u^2DU_{i+1}, \end{aligned}$$

we arrive at relation (4) for the ODE $U_{i+1} = 0$.

Let us show that L_i and J_j satisfy the conditions

$$(L_i, DJ_j) = 0, \qquad i, j \geqslant 0,\ i \neq 3k,\ j \neq 3m + 1,\ k, m = 0, 1, 2, \ldots . \tag{12}$$

For $F_1, F_2 \in A$, one has (cf. [11])

$$\begin{aligned} (F_1, DF_2) &= -(DF_1, F_2), \qquad & (F_1, P^UF_2) &= -(P^LF_1, F_2), \\ (F_1, P^JF_2) &= -(P^JF_1, F_2), \qquad & (F_1, P^LF_2) &= -(P^UF_1, F_2). \end{aligned}$$

Relations (12) follow from the formulas $(L_{-1}, DJ_j) = 0$, $(L_i, DJ_{-1}) = 0$, $i, j \geqslant 0$, and the formal chain of equations

$$\begin{aligned} (L_i, DJ_j) &= -(DL_i, J_j) = -(P^LU_{i-1}, J_j) = (U_{i-1}, P^UJ_j) = (U_{i-1}, DU_{j+1}) \\ &= -(DU_{i-1}, U_{j+1}) = -(P^UJ_{i-2}, U_{j+1}) = (J_{i-2}, P^LU_{j+1}) = (J_{i-2}, DL_{j+2}) \\ &= -(DJ_{i-2}, L_{j+2}) = -(P^JL_{i-3}, L_{j+2}) = (L_{i-3}, P^JL_{j+2}) = (L_{i-3}, DJ_{j+3}). \qquad \square \end{aligned}$$

3. Soliton solutions. As follows from (10), (11), Eq. (1) has two invariant manifolds of order $2n$ for each $n > 0$, which permits one to construct two n-soliton solutions. Application of invariant manifolds defined by the ODE (3) generalizes separation of variables for evolution equations [12]. Constructing n-soliton solutions for Eq. (1) is reduced to finding an appropriate n-parameter solution $u(x) = \varphi(x, c_1, \ldots, c_n)$ of the ODE $R_{2n}[u] = 0$ and then substituting the function $u(t, x) = \varphi(x, c_1(t), \ldots, c_n(t))$ into Eq. (1), which gives a system of first-order ODE for the parameters $c_1(t), \ldots, c_n(t)$.

The n-soliton solutions of Eq. (1) have the form

$$u(t, x) = E_n - (\ln f_n)_{xx}, \qquad E_n = \text{const}\,. \tag{13}$$

We have

$$f_1^U = \cosh a(x + c_1), \quad E_1^U = b/6, \qquad f_1^L = b + \cosh a(x + c_1), \quad E_1^L = a^2/6,$$

for the one-soliton solutions (13),

$$f_2^J = a_2 \cosh a_1(x+c_1) + b a_1 \cosh a_2(x+c_2), \qquad E_2^J = \frac{(a_1^2+3a_2^2)b^2 - 3a_1^2 - a_2^2}{18(b^2-1)},$$

$$f_2^L = \cosh a_1(x+c_1)[(2a_1^2+a_2^2)\cosh a_2(x+c_2) + b] - 3a_1a_2 \sinh a_1(x+c_1)\sinh a_2(x+c_2),$$

$$E_2^L = a_2^2/6,$$

for two-soliton solutions (13), and so on. Here a, a_i, and b are constant parameters. The superscript J, U, or L on f_n and E_n indicates that for some values of the parameters K_i and b the considered solution (13) is generated by the corresponding invariant manifold (11) of order $2n$.

References

[1] K. Sawada and T. Kotera, Prog. Theor. Phys., **51**:5 (1974), 1355–1367.
[2] P. J. Caudrey, R. K. Dodd, and J. D. Gibbon, Proc. R. Soc. Lond. A, **351**:1666 (1976), 407–422.
[3] V. E. Zakharov and L. D. Faddeev, Funkts. Anal. Prilozhen., **5**:4 (1971), 18–27; English transl.: Funct. Anal. Appl., **5**:4 (1971), 280–287.
[4] C. S. Gardner, J. Math. Phys., **12**:8 (1971), 1548–1551.
[5] J. Satsuma and D. J. Kaup, J. Phys. Soc. Japan, **43** (1977), 692–697.
[6] V. V. Sokolov and A. B. Shabat, Funkts. Anal. Prilozhen., **14**:2 (1980), 79–80; English transl.: Funct. Anal. Appl., **14**:2 (1980), 148–150.
[7] A. P. Fordy and J. Gibbons, J. Math. Phys., **21**:10 (1980), 2508–2510.
[8] F. Magri, J. Math. Phys., **19**:5 (1978), 1156–1162.
[9] I. M. Gelfand and I. Ya. Dorfman, Funkts. Anal. Prilozhen., **13**:4 (1979), 13–30; English transl.: Func. Anal. Appl., **13**:4 (1979), 248–262.
[10] B. Fuchssteiner and W. Oevel, J. Math. Phys., **23**:3 (1982), 358–363.
[11] P. Lax, SIAM Review, **18**:3 (1976), 351–375.
[12] V. K. Andreev, O. V. Kaptsov, V. V. Puckhnachov, and A. A. Rodionov, Applications of group-theoretical methods in hydrodynamics, Kluwer Academic Publishers, Netherlands, 1998.
[13] O. V. Kaptsov, Dinamika sploshnoi sredy (Institute of hydrodynamics, Siberian Branch of Academy of Sciences of USSR), **46** (1980), 46–57.

Institute of Mathematics with Computing Centre, Ufa
e-mail: yulya@mail.rb.ru

Translated by Yu. Yu. Bagderina