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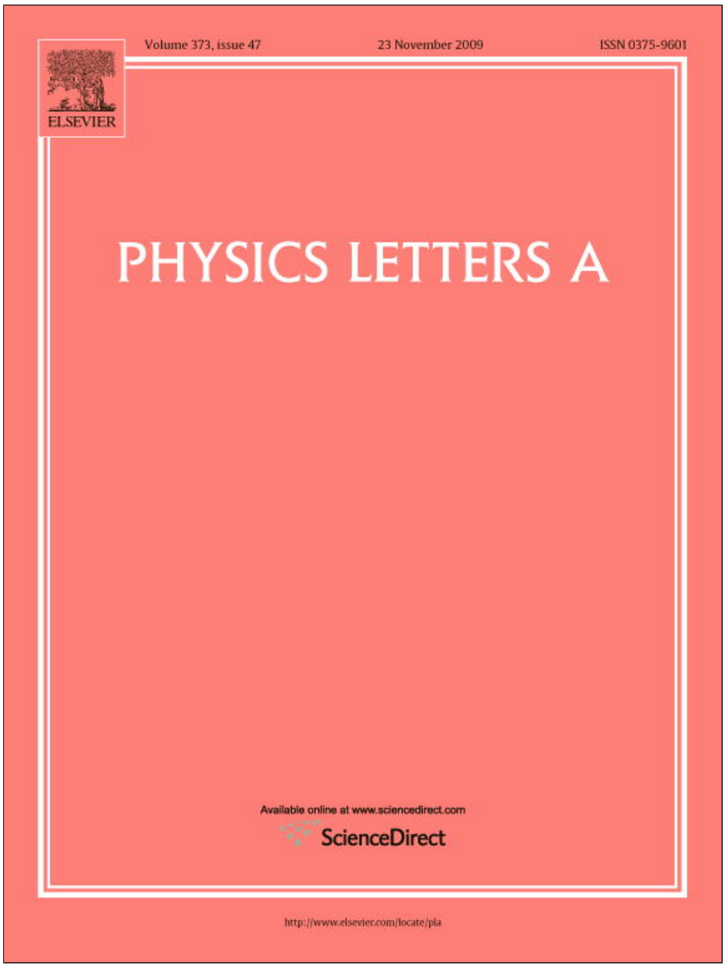

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A new family of evolution water-wave equations possessing two-soliton solutions

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ABSTRACT

We find a new family of fifth-order water-wave equations having common invariant manifold of the fourth order. These evolution equations are nonintegrable except for two cases corresponding to the Sawada–Kotera and Kaup–Kupershmidt equations. The invariant manifold of the family is an autonomous equation F-VI from the Cosgrove's classification of fourth-order ODEs having the Painlevé property. Two-parameter solutions of the equation F-VI allow to find two-soliton solutions for this family of evolution equations.

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1. Introduction

As is well known, the Kaup–Kupershmidt (KK) equation [1]

$$v_t = v_{xxxxx} - 15vv_{xxx} - \frac{75}{2}v_x v_{xx} + 45v^2 v_x \tag{1}$$

and the Sawada–Kotera (SK) equation [2,3]

$$v_t = v_{xxxxx} - 30vv_{xxx} - 30v_x v_{xx} + 180v^2 v_x \tag{2}$$

have an invariant manifold [3,4]

$$v_{xxxx} - 18vv_{xx} - 9v_x^2 + 24v^3 + K_2\bigl(v_{xx} - 3kv^2\bigr) + K_1 = 0,$$

where $k = 2$ for the KK equation and $k = 1$ for the SK equation, K_1, K_2 are arbitrary constants. Translation $v = u + K_2/18$ transforms it into an autonomous case (with $\alpha = (-1)^k kK_2$, $\beta = (6K_2^3 + \alpha^3)/972 - K_1$)

$$u_{xxxx} = 18uu_{xx} + 9u_x^2 - 24u^3 + \alpha u^2 + \frac{\alpha^2}{9}u + \beta \tag{3}$$

of the equation F-VI from the classification of Cosgrove [5] of the fourth-order ordinary differential equations (ODEs) having the Painlevé property. Other ODEs in this classification, the equations F-III, F-IV and autonomous equation F-V, represent the group-invariant reduction of the KK, SK and fifth-order Korteweg–de Vries (KdV) equations respectively. In [5] the general solution of equations F-III, ..., F-VI has been constructed in terms of hyperelliptic functions of genus two. Particular two-parameter solutions of these ODEs allow to find two-soliton solutions for evolution equations of the corresponding integrable hierarchies.

According to [6], an ODE

$$\Phi\bigl(x, u, u_x, \dots, \partial^k u/\partial x^k\bigr) = 0 \tag{4}$$

defines the invariant manifold of the evolution equation

$$u_t = F\bigl(x, u, u_x, \dots, \partial^p u/\partial x^p\bigr), \tag{5}$$

if the relation

$$X\Phi|_{[\Phi]=0} = 0, \qquad X = F\partial_u + \sum_{j=1}^{k} D_x^j F\partial_{u_j} \tag{6}$$

holds. Here $u_j = \partial^j u/\partial x^j$, D_x is the operator of total differentiation with respect to x, $[\Phi] = 0$ is the manifold defined by Eq. (4) and its consequences $D_x^j\Phi = 0$, $j = 1, \dots, p$. If the solution

$$u(x) = U(x, c_1, \dots, c_k), \quad c_1, \dots, c_k = \text{const}$$

of ODE (4) is found, then evolution equation (5) have a solution of the form

$$u(t, x) = U\bigl(x, c_1(t), \dots, c_k(t)\bigr).$$

The substitution of this function into Eq. (5) yields the system of first-order ODEs in $c_1(t), \dots, c_k(t)$. Thus, finding the exact solutions to evolution equations with the use of invariant manifolds (4) can be regarded as a generalized separation of variables.

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In [7,8] the linear invariant manifolds were applied to constructing the solutions of the form of finite sums

$$u(t,x) = c_1(t) f_1(x) + c_2(t) f_2(x) + \cdots + c_k(t) f_k(x).$$

But most of the solutions of physical interest (N-soliton, rational, etc.) have the form

$$u(t,x) = \frac{c_0(t) f_0(x) + c_1(t) f_1(x) + \cdots + c_k(t) f_k(x)}{c_0(t) g_0(x) + c_1(t) g_1(x) + \cdots + c_k(t) g_k(x)}.$$

To find them, in a recent paper [9] we have proposed to use the nonlinear invariant manifolds (4) depending on u, $u_x, \ldots, \partial^k u/\partial x^k$ like the generalized Riccati equations [10]. This approach was illustrated with the equation

$$v_t + v_{xxxxx} - a v v_{xxx} + 2(a-30) v_x v_{xx} + v_{xxx} - c v v_x = 0, \quad a, c = \text{const}, \tag{7}$$

which describes the long waves in shallow water [11]. It has been proved in [12] that Eq. (7) is nonintegrable, it does not pass the WTC-test [13]. We have found [9] that for $a = 10$ Eq. (7) possesses an invariant manifold

$$v_{xxxx} - 18 v v_{xx} - 9 v_x^2 + 24 v^3 + \frac{9-c}{5} v_{xx} + \frac{9}{10}(c-8) v^2 + (c-6)(c-12)\frac{v}{100} + K_1 = 0.$$

Translation $v = u + (9-c)/90$ turns this ODE into Eq. (3) with $\alpha = -c/10$, $\beta = 3(c-9)(8c^2 - 171c + 648)/90^3 - K_1$. Hence there exist three evolution equations (1), (2) and (7) having the invariant manifolds reduced to the same equation F-VI.

Then an inverse problem arises: to describe all evolution equations of the form

$$u_t = u_{xxxxx} + A_1 u u_{xxx} + A_2 u_x u_{xx} + A_3 u^2 u_x + A_4 u_{xxx} + A_5 u u_x, \quad A_1, \ldots, A_5 = \text{const}, \tag{8}$$

having the invariant manifolds, which coincide with ODEs F-III, ..., F-VI from [5]. Eq. (8) admits five-parameter equivalence group of time and space translations, two dilations and a Galilean transformation

$$\tau = a_3^{-5} t + a_5, \qquad z = a_3^{-1}\big(x + (a_1^2 A_3 - a_1 A_5)t\big) + a_4, \quad v = a_3^2 a_2^{-1}(u + a_1), \quad a_2, a_3 \neq 0, \tag{9}$$

which turn Eq. (8) into an equation of the same form

$$v_\tau = v_{zzzzz} + a_2 A_1 v v_{zzz} + a_2 A_2 v_z v_{zz} + a_2^2 A_3 v^2 v_z + a_3^2 (A_4 - a_1 A_1) v_{zzz} + a_3^2 a_2 (A_5 - 2a_1 A_3) v v_z.$$

Relation (6) for ODE (3) and evolution equation (8) becomes

$$2(54A_4 + 5A_5 + 10\alpha) u_{xx} u_{xxx} + \big(342A_4 + 27A_5 + \alpha(8A_1 + 2A_2 + 270)\big) u_x^3 - 120(A_1 + 2A_2 + 90) u^3 u_{xxx} - 240(9A_1 + A_3 + 90) u^4 u_x + \cdots = 0$$

(here we omit the remaining terms of the polynomial) and vanishes with $A_1 = -A_3/9 - 10$, $A_2 = A_3/18 - 40$, $A_4 = 5\alpha \times (A_3/18 - 4)/18$, $A_5 = \alpha(10 - A_3/6)$. Hence equation F-VI provides the invariant manifold for a whole family of evolution equations

$$u_t = u_{xxxxx} - 10(b+1) u u_{xxx} + 5(b-8) u_x u_{xx} + 90 b u^2 u_x + \frac{5}{18}\alpha(5b-4) u_{xxx} + 5\alpha(2-3b) u u_x, \quad b = \text{const}. \tag{10}$$

Similar computation shows that equations F-III, F-IV and F-V define the invariant manifold only for the KK, SK and KdV equations respectively. Solving these ODEs one can find traveling wave solutions of the corresponding fifth-order evolution equations.

Eq. (10) coincides (up to transformation (9), where $a_1 = \alpha(1-b)/18$, $a_2 = 1$, $a_3 = 1$, $a_4 = 0$, $a_5 = 0$) with the KK equation (1) in the case of $b = 1/2$ and with the SK equation (2), when $b = 2$. It is nonintegrable for other values of b. In [14] WTC-type expansions with logarithmic terms have been constructed for a class of equations (8). It was established there that the solutions of 27 equations of the form (8) can be expanded into the series with or without logarithms with four nonnegative resonances. Seven of these equations are involved into the family (10), when b equals to 2, 1/2, 2/5, 4/5, 4, 13/10 or 29/10. The class of evolution equations (8) occurs in water-wave models and several other applications (see [15] for many references).

It was shown in [16] for Eq. (8) with the parameters $A_1 = 8A - 2B$, $A_2 = 4A - 6B$, $A_3 = -20AB$, $A_4 = 0$, $A_5 = 0$ that three cases when it is a soliton equation (SK, KK or KdV) are closely related to the only cases when the cubic Hénon–Heiles Hamiltonian system passes the Painlevé test. This equation

$$v_\tau = \left(v_{zzzz} + (8A - 2B) v v_{zz} - 2(A+B) v_z^2 - \frac{20}{3} A B v^3\right)_z \tag{11}$$

is related to Eq. (10) with $\alpha = 0$ by transformation (9), iff the equalities

$$10a_2(b+1) = 2B - 8A, \qquad 5a_2(b-8) = 4A - 6B, \qquad 90a_2^2 b = -20AB$$

hold. With $4A = a_2(2-7b)$, $B = a_2(7-2b)$ obtained from first two equalities the third one becomes $a_2^2(b-2)(2b-1) = 0$. Therefore, integrable SK and KK equations are the only equations (8) involved into both families (10) and (11).

The outline of this Letter is as follows. In Section 2 other nonlinear invariant manifolds of Eq. (10) are found from criterion (6). Using the solutions of these ODEs, the traveling wave solutions to Eq. (10) are constructed. In Section 3 we study two-soliton and oscillating solutions of Eq. (10) arising from two-parameter solutions of the equation F-VI. In Section 4 we construct bilinear form of Eq. (10) following the approach applied in [17] to the KK equation. Next we show that two-soliton solution obtained in Section 3 may be found also using Hirota's bilinear method [18].

2. Traveling wave solutions of Eq. (10)

Invariant manifolds of Eq. (10) are found from the criterion (6), where we set the right-hand side of Eq. (10) in place of F. It turns out that, in addition to ODE (3), Eq. (10) has the following nonlinear invariant manifolds:

$$u_{xx} - 6u^2 - \frac{\alpha}{3} u + K_1 = 0, \tag{12}$$

$$u_{xx} - \frac{3}{2} b u^2 + K_2 u + K_1 = 0, \quad K_2 = \alpha \frac{5b^2 - 22b + 12}{12(b-5)}, \quad b \neq 5; 0, \tag{13}$$

$$(u - K_2)\left(u_{xx} - 3u^2 + \frac{\alpha}{3} u + K_1\right) - \frac{3}{4} u_x^2 = 0, \quad K_2 = \frac{5}{36}\alpha, \tag{14}$$

$$u_{xxxx} - 10(b+1) u u_{xx} + \frac{15}{2}(b-2) u_x^2 + 30 b u^3 + \frac{5}{18}\alpha(5b-4) u_{xx} + \frac{5}{2}\alpha(2-3b) u^2 + \tilde{K}_2 u + \tilde{K}_1 = 0, \quad K_1, \tilde{K}_1, \tilde{K}_2 = \text{const}. \tag{15}$$

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The last equation describes the traveling wave solutions $u(t,x) = u(x - \tilde{K}_2(t+t_0))$ of Eq. (10). Note that differentiated once ODEs (3), (12), (13), (15) define also the invariant manifolds. Eqs. (12) and (13) coincide for $b = 4$. When $b = 2$ or $b = 1/2$ evolution equation (10) is integrable and has infinitely many invariant manifolds [4], but we will not consider them here.

We have checked that (12) is identical to the first auxiliary variable H used in [5] for integrating the equation F-VI by means of Jacobi's postmultiplier theory. But considering Eq. (13) instead of the second auxiliary variable J used in [5] does not seem to shorten the integration as performed by Cosgrove.

Integrating ODEs (12)–(14), one obtains

$$u_x^2 = 4u^3 + \frac{\alpha}{3}u^2 - 2K_1 u + K_0, \tag{16}$$

$$u_x^2 = bu^3 - K_2 u^2 - 2K_1 u + K_0, \tag{17}$$

$$u_x^2 = 4(u - K_2)\left(u^2 + \frac{2}{9}\alpha u - \frac{5}{81}\alpha^2 + K_1\right) + 4K_0(u - K_2)^{3/2}, \tag{18}$$

$K_0 = \text{const}$. ODE (18) with $u = w^2 + K_2$ becomes

$$w_x^2 = w^4 + \frac{3}{2}\alpha w^2 + K_0 w + K_1 - \frac{5}{48}\alpha^2.$$

Therefore the solutions of all ODEs (16)–(18) are expressible in terms of elliptic functions. Moreover, Eqs. (16)–(18) represent the integrals also for ODE (15) with the relevant parameters $\tilde{K}_1$, $\tilde{K}_2$. For appropriately chosen K_0, K_1 ODEs (16)–(18) are solved in terms of hyperbolic functions, which implies the following solitary wave solutions of evolution equation (10):

$$u(t,x) = m^2\left(\tanh^2 m(x+c_1) - 2/3\right) - \alpha/36,$$
$$c_1'(t) \equiv C_1 = 2m^4(4-5b)/3 + 5\alpha^2(7b-4)/72, \tag{19}$$

$$u(t,x) = \frac{4}{b}m^2\left(\tanh^2 m(x+c_1) - 2/3\right) + \frac{K_2}{3b},$$
$$c_1'(t) \equiv \hat{C}_1 = \frac{16m^4}{3b}(20-7b) - 5\alpha\frac{b^2-12b+8}{6b(b-5)}K_2, \tag{20}$$

$$u(t,x) = \frac{\alpha}{18} + \frac{m^2}{6} - \frac{m^2(M\cosh m(x+c_1)+1)}{(\cosh m(x+c_1)+M)^2},$$
$$M^2 = \frac{\alpha - 2m^2}{\alpha + m^2},$$
$$c_1'(t) \equiv \bar{C}_1 = m^4(5b-4)/6 + 5\alpha^2(1-b)/9. \tag{21}$$

The last solution has no singular points, if $m^2 < \alpha/2$ in the case of $\alpha > 0$ or if $m^2 < |\alpha|$, $M > 0$, when $\alpha < 0$. The ODEs in $c_1(t)$ arise when we substitute solutions (19)–(21) with $c_1 = c_1(t)$ into Eq. (10). These equations of the form $c_1'(t) = \text{const}$ are readily integrated to obtain $c_1(t) = C_1(t+t_0)$, etc. Substituting

$$m = i\tilde{m}, \qquad \sinh m(x+c_1) = i\sin\tilde{m}(x+c_1),$$
$$\cosh m(x+c_1) = \cos\tilde{m}(x+c_1) \tag{22}$$

into formulas (19)–(21), one obtains periodic solutions of the evolution equation (10). Only one of them, namely, the solution

$$u(t,x) = \frac{\alpha}{18} - \frac{\tilde{m}^2}{6} + \frac{\tilde{m}^2(\tilde{M}\cos\tilde{m}(x+c_1)+1)}{(\cos\tilde{m}(x+c_1)+\tilde{M})^2},$$
$$\tilde{M}^2 = \frac{\alpha + 2\tilde{m}^2}{\alpha - \tilde{m}^2}$$

is free of singularities when $\alpha > 0$, $\tilde{m}^2 < \alpha$.

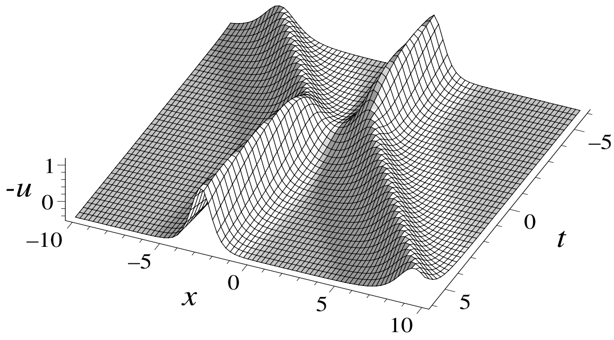

Fig. 1. Two-soliton solution of Eq. (10): $m = 9/7$, $n = 4/7$, $b = 1$, $\alpha = 4$.

3. Two-phase solutions of Eq. (10)

In [5] the equation F-VI has been solved in terms of hyperelliptic functions of genus two. In particular cases ODE (3) have two-parameter solutions expressible in terms of hyperbolic, trigonometric or power functions. The common form of these solutions is

$$u(x, c_1, c_2) = \frac{m^2}{3} - \frac{\alpha}{36} - (\ln f)_{xx}. \tag{23}$$

For different types of solutions (23) the function f is defined as follows.

3.1. Two-soliton solution

If the parameters m, n satisfy the condition

$$4m^2 - 8n^2 = \alpha, \tag{24}$$

then we have

$$f = \cosh m(x+c_1)\left[(m^2-n^2)\cosh^2 n(x+c_2) + 3n^2\sinh^2 n(x+c_2)\right] - 3mn\sinh m(x+c_1)\sinh n(x+c_2)\cosh n(x+c_2). \tag{25}$$

Solution (23) with f given by (25) has no singular points provided $m > 2n$. Substitution of this solution with $c_1 = c_1(t)$, $c_2 = c_2(t)$ into the evolution equation (10) leads to the relations

$$c_1'(t) \equiv C_1,$$
$$c_2'(t) \equiv C_2 = \frac{8}{3}n^4(5b-4) + \frac{5}{9}\alpha^2(1-b) \tag{26}$$

with C_1 given in (19). Integrating (26), one obtains $c_1(t) = C_1(t+t_1)$, $c_2(t) = C_2(t+t_2)$, $t_1, t_2 = \text{const}$. Fig. 1 represents two-soliton collision ($m = 9/7$, $n = 4/7$) for Eq. (10) with the parameters $b = 1$, $\alpha = 4$ (we plot the physical wave $-u(t,x)$). It typically shows the two 'solitary' waves entering, and then emerging from the interaction intact (except for the phase shifts which are clearly visible). Since

$$\frac{f_x}{f} = \frac{(4n^2 - m^2)f_0}{m^2 - n^2 + 3n\tanh n(x+c_2) f_0},$$
$$f_0 = n\tanh n(x+c_2) - m\tanh m(x+c_1),$$

it is not difficult to calculate at every moment t the total area A between the curve $u(t,x)$ and the line $u = m^2/3 - \alpha/36$

$$A = \int_{-\infty}^{\infty} (\ln f)_{xx}\,dx = \left[\frac{f_x}{f}\right]_{x=-\infty}^{x=\infty} = 2(m+2n).$$

Hence the value of A is constant for all time.

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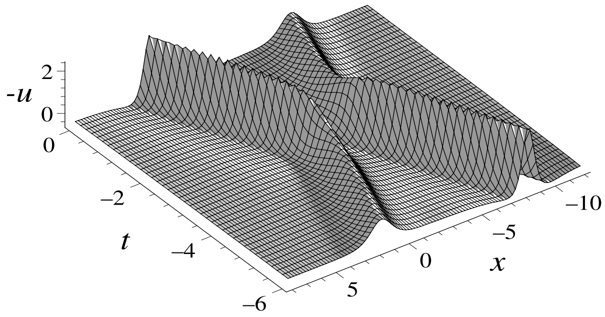

Fig. 2. Solution (23), (27) of Eq. (10): $m = 1$, $b = 1$, $\alpha = 4$.

3.2. Two-soliton-like solution

If $\alpha > 0$ in Eq. (10), we have also solution (23) with the function

$$f = \left(m^2(x+c_2)^2 + 3\right)\cosh m(x+c_1) - 3m(x+c_2)\sinh m(x+c_1), \tag{27}$$

where the parameter m is given by

$$4m^2 = \alpha. \tag{28}$$

Substitution of solutions (23), (27) with $c_1 = c_1(t)$, $c_2 = c_2(t)$ into Eq. (10) leads to relations (26), where $n \equiv 0$. As is seen from Fig. 2, this solution (with $m = 1$) behaves much like a previous solution of Eq. (10). In this case

$$\frac{f_x}{f} = \frac{m^3(x+c_2)^2 \tanh m(x+c_1) - m^2(x+c_2)}{m^2(x+c_2)^2 + 3 - 3m(x+c_2)\tanh m(x+c_1)}$$

and one can readily obtain

$$A = \int_{-\infty}^{\infty} (\ln f)_{xx}\,dx = \left[\frac{f_x}{f}\right]_{x=-\infty}^{x=\infty} = 2m.$$

3.3. Oscillating solution

If $\alpha > 0$ in Eq. (10), then substituting

$$n = i\tilde{n}, \qquad \sinh n(x+c_2) = i\sin\tilde{n}(x+c_2),$$
$$\cosh n(x+c_2) = \cos\tilde{n}(x+c_2)$$

into the formula (25), one can find an oscillating solution (23) with the function f given by

$$f = \cosh m(x+c_1)\left[\left(m^2+\tilde{n}^2\right)\cos^2\tilde{n}(x+c_2) + 3\tilde{n}^2\sin^2\tilde{n}(x+c_2)\right] + 3m\tilde{n}\sinh m(x+c_1)\sin\tilde{n}(x+c_2)\cos\tilde{n}(x+c_2), \tag{29}$$

where the parameters m, $\tilde{n}$ satisfy the condition

$$4m^2 + 8\tilde{n}^2 = \alpha. \tag{30}$$

Substitution of this solution into the evolution equation (10) leads to the relations (26) with $\tilde{n}$ instead of n.

Substitution of (22) into functions (25), (27), (29) provides other three solutions of the form

$$u(x, c_1, c_2) = -\frac{\tilde{m}^2}{3} - \frac{\alpha}{36} - (\ln f)_{xx}$$

to Eq. (10), but all of them will have singular points.

Note that conditions (24), (28) or (30) are not imposed on solution (23) of Eq. (10) when $b = 2$ or $b = 1/2$. In these integrable cases the parameters m, n, $\tilde{n}$ in functions (25), (27), (29) are independent. This conforms to Hietarinta's assertion that any fifth-order PDE with suitable nonlinearity can possess two-soliton solution, but if this PDE is nonintegrable, then the parameters of the solution are restricted.

4. Bilinear form and solitary wave

Here we construct bilinear form of Eq. (10). Following Hirota's method [18], we make a change of dependent variable

$$u = \gamma\partial_x^2 \ln F(t,x), \quad \gamma = \text{const}. \tag{31}$$

Eq. (10) can be integrated once with respect to x to give

$$-\gamma(\ln F)_{xt} + H(u) = \mu, \tag{32}$$

where we have made use of Eq. (31) and

$$H(u) = u_{xxxx} - 10(b+1)uu_{xx} + \frac{15}{2}(b-2)u_x^2 + 30bu^3 + \frac{5}{18}\alpha(5b-4)u_{xx} + \frac{5}{2}\alpha(2-3b)u^2.$$

The arbitrary function of integration has been set to constant μ in Eq. (32) by imposing nonzero boundary conditions $u \to \text{const}$ as $x \to \pm\infty$ (since we know that solution (23), (25) decays to constant $m^2/3 - \alpha/36$ as $x \to \pm\infty$). The first term of (32) can be cast into bilinear form by noting the identity

$$(\ln F)_{xt} = F^{-2}D_xD_tF\cdot F/2, \tag{33}$$

where D_x, D_t are the Hirota derivatives [18] defined by

$$D_x^aD_t^bF(t,x)\cdot G(t,x) = (\partial_x - \partial_{x'})^a(\partial_t - \partial_{t'})^bF(t,x)G(t',x')\big|_{x'=x,t'=t}.$$

Family (10) involves the KK equation which has been bilinearized in [17]. To bilinearize Eq. (10) we proceed in a similar way. First let us note the identities for the function (31) (cf. [17])

$$\gamma^2F^{-2}\left(\gamma D_x^6F\cdot F + kD_x^4F\cdot F\right) = 2\gamma^2u_{xxxx} + 60\gamma uu_{xx} + 120u^3 + k\left(2\gamma u_{xx} + 12u^2\right), \tag{34}$$

$$\gamma^2F^{-2}D_x^2F\cdot G = 2\gamma uG/F + \gamma^2(G/F)_{xx}, \tag{35}$$

$$\gamma u\left(\gamma F^{-2}D_x^4F\cdot F + lF^{-2}D_x^2F\cdot F\right) = 2\gamma uu_{xx} + 12u^3 + 2lu^2, \tag{36}$$

$$\gamma^2\partial_x^2\left(\gamma F^{-2}D_x^4F\cdot F + lF^{-2}D_x^2F\cdot F\right) = 2\gamma^2u_{xxxx} + 24\gamma uu_{xx} + 24\gamma u_x^2 + 2l\gamma u_{xx}, \tag{37}$$

where $k, l = \text{const}$, $G(t,x)$ is an arbitrary function. If we define G/F by

$$G/F = \left(\gamma F^{-2}D_x^4F\cdot F + lF^{-2}D_x^2F\cdot F\right)/2, \tag{38}$$

then relations (35)–(37) imply

$$\gamma^2F^{-2}D_x^2F\cdot G = \gamma^2u_{xxxx} + \gamma\left(14uu_{xx} + 12u_x^2\right) + 12u^3 + l\left(\gamma u_{xx} + 2u^2\right). \tag{39}$$

Suppose that the relation

$$H(u) = \kappa\gamma^2F^{-2}\left(\gamma D_x^6F\cdot F + kD_x^4F\cdot F\right) + \lambda\gamma^2F^{-2}D_x^2F\cdot G \tag{40}$$

holds for some constants κ, λ and substitute (34), (39) into (40). Equality (40) is possible only if

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$$\gamma=-1,\qquad \kappa=\frac{1}{16}(5b-2),\qquad \lambda=\frac{5}{8}(2-b),$$
$$\kappa k=\frac{5}{144}\alpha(10-17b),\qquad l=\frac{\alpha}{3}.$$

Then relations (32), (33), (38), (40) provide the bilinear representation of Eq. (10)

$$\begin{aligned}&\big[8D_xD_t+5\alpha(10-17b)/9D_x^4+(2-5b)D_x^6\big]F\cdot F\\&\quad+10(2-b)D_x^2F\cdot G=16\mu F\cdot F,\end{aligned}\tag{41}$$
$$\big[D_x^4-\alpha/3D_x^2\big]F\cdot F+2F\cdot G=0,\tag{42}$$

with $u=-(\ln F)_{xx}$. If $b=2$ then $\kappa=1/2$, $\lambda=0$ and Eqs. (41), (42) become decoupled. For $\alpha=0$, $\mu=0$ this yields well known bilinear form $(D_xD_t-D_x^6)F\cdot F=0$ of the SK equation [3]. The coupled bilinear form of the SK and KK equations can be found in [17,19].

Let us show that the solitary wave (21) solves the bilinear system (41), (42). Function (21) has the form $u=-(\ln F)_{xx}$ with

$$F=\mathrm{e}^{\varepsilon x^2}\big(1+\mathrm{e}^{\theta}+M^{-2}\mathrm{e}^{2\theta}/4\big),\tag{43}$$

where $\theta=mx+\omega t+\eta$, ω, $\eta=\text{const}$, $\varepsilon=-(3m^2+\alpha)/36$. Substituting (43) into Eq. (42) yields the function

$$G=2\varepsilon\big(m^2+2\alpha/3\big)F+m^2\big(m^2+\alpha\big)\mathrm{e}^{\varepsilon x^2}\mathrm{e}^{\theta}.\tag{44}$$

Functions F and G solve Eq. (41) provided that $\mu=20\varepsilon^2(bm^2+\alpha(1-7b/6))$, $\omega=m\bar{C}_1$ with $\bar{C}_1$ given in (21). Comparing functions (43), (44) and ansatz $F=1+\mathrm{e}^{\theta}+\mathrm{e}^{2\theta}/16$, $G=-m^4\mathrm{e}^{\theta}$ used in [17] for the 'anomalous' solitary wave of the KK equation (1), it is readily seen that they differ by terms involving ε.

5. Direct method of finding two-soliton solution

An observation on the solitary waves made in previous section prompts to supplement Parker's ansatz for two-soliton solution of the KK equation [20] by terms with the parameter ε as

$$\begin{aligned}F&=\mathrm{e}^{\varepsilon x^2}\big(1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+k_1\mathrm{e}^{2\theta_1}+k_2\mathrm{e}^{\theta_1+\theta_2}+k_3\mathrm{e}^{2\theta_2}+k_4\mathrm{e}^{2\theta_1+\theta_2}\\&\quad+k_5\mathrm{e}^{\theta_1+2\theta_2}+k_6\mathrm{e}^{2(\theta_1+\theta_2)}\big),\end{aligned}\tag{45}$$
$$\begin{aligned}G&=2\varepsilon j_1F+\mathrm{e}^{\varepsilon x^2}\big(j_2\mathrm{e}^{\theta_1}+j_3\mathrm{e}^{\theta_2}+j_4\mathrm{e}^{\theta_1+\theta_2}\\&\quad+j_5\mathrm{e}^{2\theta_1+\theta_2}+j_6\mathrm{e}^{\theta_1+2\theta_2}\big),\end{aligned}$$

where $\theta_i=p_ix+\omega_it+\eta_i$, $i=1,2$ and $k_1,\dots,k_6$, $j_1,\dots,j_6$ are constant coefficients. Substituting functions F and G into Eq. (42), we find that

$$\varepsilon=\frac{\alpha}{72}+\frac{p_1^2(1-16k_1)}{24(4k_1-1)},$$
$$k_2=\frac{(p_1^2-p_2^2)(p_1^2-4p_2^2)+8k_1(p_1^2+2p_2^2)(4p_1^2-p_2^2)}{(p_1+p_2)^2((p_1+2p_2)^2+16k_1(2p_1^2-p_1p_2-p_2^2))},$$
$$k_3=\frac{p_1^2-p_2^2+4k_1(p_2^2-4p_1^2)}{4(p_1^2-4p_2^2+16k_1(p_2^2-p_1^2))},\qquad k_4=k_1k_0,$$
$$k_5=k_3k_0,\qquad k_6=k_1k_3k_0^2,$$
$$j_1=\alpha/3-12\varepsilon,\qquad j_2=p_1^2\big(j_1-12\varepsilon-p_1^2\big),$$
$$j_3=p_2^2\big(j_1-12\varepsilon-p_2^2\big),$$
$$\begin{aligned}j_4&=2p_1p_2\big(2p_1^2-3p_1p_2+2p_2^2-j_1+12\varepsilon\big)\\&\quad+k_2(p_1+p_2)^2\big(j_1-12\varepsilon-(p_1+p_2)^2\big),\end{aligned}$$
$$j_5=k_4j_3,\qquad j_6=k_5j_2,$$

where

$$k_0=\frac{(p_1-p_2)^2((p_1-2p_2)^2+16k_1(2p_1^2+p_1p_2-p_2^2))}{(p_1+p_2)^2((p_1+2p_2)^2+16k_1(2p_1^2-p_1p_2-p_2^2))}.$$

Substitution of functions F, G with these parameters into (41) leads to

$$\mu=10\varepsilon^2\alpha(2-3b)-240\varepsilon^3b,$$
$$\begin{aligned}\omega_i&=p_i^5+5p_i^3\big(\alpha(5b-4)/18+4\varepsilon(b+1)\big)\\&\quad+10p_i\varepsilon\big(\alpha(3b-2)+36\varepsilon b\big),\quad i=1,2,\end{aligned}$$

and a number of relations on k_1. The simplest of them is given by

$$(2b-1)k_1\big(p_1^2(8k_1+1)+\alpha(1-4k_1)\big)=0.\tag{46}$$

If $b=1/2$, then all relations on k_1 become identities and one obtains two-soliton solution of the KK equation.

If we take $k_1=0$, then all remaining relations are satisfied provided that the condition $p_1^2-2p_2^2=\alpha$ holds (cf. (24)). Then function (45) may be written as

$$\begin{aligned}F&=\frac{\mathrm{e}^{\varepsilon x^2}}{(p_1-p_2)(p_1-2p_2)}\big[\big(p_1^2+2p_2^2-3p_1p_2\big)\big(1+\mathrm{e}^{\bar\theta_1+2\bar\theta_2}\big)\\&\quad+\big(p_1^2+2p_2^2+3p_1p_2\big)\big(\mathrm{e}^{\bar\theta_1}+\mathrm{e}^{2\bar\theta_2}\big)\\&\quad+2\big(p_1^2-4p_2^2\big)\big(\mathrm{e}^{\bar\theta_1+\bar\theta_2}+\mathrm{e}^{\bar\theta_2}\big)\big],\end{aligned}$$

where $\bar\theta_1=\theta_1+\ln k_2$, $\bar\theta_2=\theta_2+\ln(p_1-p_2)(p_1+2p_2)^{-1}/2$. It is now straightforward to check that the function $u=-(\ln F)_{xx}$ will coincide with solution (23), (25) of Eq. (10), if we set $p_1=2m$, $p_2=2n$.

Note that the choice of $k_1=(\alpha+p_1^2)(\alpha-2p_1^2)^{-1}/4$ in relation (46) will lead to the same result up to replacement $\theta_1\leftrightarrow\theta_2$.

Thus, we have known the form of solitary wave (21) and two-soliton solution (23), (25), which have been found using invariant manifolds (14), (3) of Eq. (10). Therefore, we can improve Parker's approach by introducing two parameters: μ in bilinear form (41), (42) and ε in ansatz (43) and (45), and then recover these solutions using Hirota's method.

6. Conclusion

An invariant manifold is a more general object than the symmetry or the variational derivative of conserved density. Both of them represent particular cases of invariant manifold. Finding the exact solutions to an evolution equation with the use of invariant manifolds reduces the problem to the successive integrations of a higher order ODE with respect to x and then of a system of first-order ODEs with respect to t. Therefore, it is natural to speak in this case about the generalized separation of variables for an evolution equation.

In Sections 2 and 3 we have shown how this method can be applied to find explicit solutions to evolution water-wave equations (10). We have found one- and two-soliton solutions of this family of equations, using the nonlinear invariant manifolds of a special form (higher order Riccati equations).

Note that when we apply nonlinear invariant manifolds, the resulting system of first-order ODEs for the parameters $c_i(t)$ consists of linear uncoupled equations. In the case of linear manifolds such a system has more complicated form. As a rule, it is a system of coupled nonlinear ODEs [7,8]. It means that nonlinearity of the evolution equation (5) does not disappear when one applies the linear manifolds (4). Nonlinearity occurs either in invariant manifold (4) or in the system of ODEs for the functions $c_i(t)$.

Linear invariant manifolds have been studied in detail over the last twenty years. However, their application in finding the exact solutions is efficient mainly for diffusion equations. The upper

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bound on the order of linear invariant manifolds is known [8]. Given an evolution equation (5) of order p, the order k of its invariant manifold defined by linear ODE (4) satisfies the inequality $k \leqslant 2p + 1$.

The case of nonlinear invariant manifolds is quite different. Integrable equations (e.g., KK, SK, KdV equations) have no limit for the order k, they can possess invariant manifolds (4) of any order. In the case of nonintegrable evolution equations often we have $k \leqslant p$. Moreover, for the most part these equations can possess just one invariant manifold, which describes the traveling wave solutions of the equation. But certain of the nonintegrable evolution equations (as Eq. (10), for example) have another invariant manifolds, in addition to the traveling wave reductions. This feature allows to find new interesting solutions to these equations.

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