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ORDINARY DIFFERENTIAL EQUATIONS

Equivalence of Ordinary Differential Equations $y'' = R(x, y)y'^2 + 2Q(x, y)y' + P(x, y)$

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1. INTRODUCTION

Reduction of nonlinear equations of mathematical physics (the Boussinesq, Korteweg–de Vries, and sine-Gordon equation, etc.) often leads to the integration of an ordinary differential equation of the form [1, pp. 34, 52, 72, 100]

$$y'' = R(x, y)y'^2 + 2Q(x, y)y' + P(x, y). \tag{1}$$

The same class contains 50 equations whose general solutions have no movable critical points [2, Chap. 14]. There is a conjecture that a nonlinear partial differential equation can be integrated by the inverse scattering method if its arbitrary reduction leads (possibly, after a change of variables) to an ordinary differential equation with this property [3]. However, after one has obtained an equation of the form (1), it may prove difficult to find a transformation relating this equation to a canonical equation of the same class. This problem can be solved with the use of invariants of equivalence transformations of Eq. (1).

An equivalence transformation of Eq. (1) has the form

$$\bar{x} = \varphi(x), \qquad \bar{y} = \psi(x, y) \tag{2}$$

and reduces Eq. (1) to an equation of the same form,

$$\bar{y}'' = \bar{R}\,(\bar{x}, \bar{y})\,\bar{y}'^2 + 2\bar{Q}\,(\bar{x}, \bar{y})\,\bar{y}' + \bar{P}\,(\bar{x}, \bar{y})\,. \tag{3}$$

By virtue of the approach in [4], Eqs. (1) and (3) are equivalent [there exists a transformation (2) relating these equations] if and only if their invariants are equal.

The group of equivalence transformations (2) has infinitely many invariants, which are functions depending on x, y, P, Q, and R and the derivatives of P, Q, and R with respect to x and y. The highest order of the derivatives occurring in an invariant specifies the order of the invariant. The set of invariants always contains a finite subset referred to as a basis [5, Sec. 24.8]. Higher-order invariants can be derived from basis invariants by application of invariant differentiation operators $\mathscr{D}$, which have the following property: if I is a group invariant, then $\mathscr{D}I$ is also an invariant.

In the present paper, we use the symmetry approach [6] to construct a basis of invariants of Eq. (1) and find the corresponding invariant differentiation operators, i.e., solve the problem on the equivalence of equations of the class (1) under point transformations (2). If a reduction of a partial differential equation leads to an ordinary differential equation of the more general form

$$y'' = S(x, y)y'^3 + 3R(x, y)y'^2 + 3Q(x, y)y' + P(x, y), \tag{4}$$

then, to pass to an equation of the form (3), it suffices to make the change of variables

$$\bar{x} = \chi(x, y), \qquad \bar{y} = \psi(x, y), \tag{5}$$

where $\chi(x,y)$ is a function satisfying the relation

$$S\chi_x^3 - 3R\chi_x^2\chi_y + 3Q\chi_x\chi_y^2 - P\chi_y^3 - \chi_y^2\chi_{xx} + 2\chi_x\chi_y\chi_{xy} - \chi_x^2\chi_{yy} = 0$$

[for example, a hodograph transformation if $P(x,y) = 0$ in Eq. (4)].

The problem on the equivalence of second-order ordinary differential equations to specific types of equations was studied in [7–10]. The paper [7] gives a criterion for the equivalence of second-order ordinary differential equations to the first two Painlevé equations under transformations (2). The problem on the equivalence of Eq. (1) of the special form $y'' = P(x,y)$ to six Painlevé equations was solved in [8]. Necessary and sufficient conditions for the linearization of second-order ordinary differential equations by point transformations (5) were proved in [9]. Problems on the equivalence of equations of the class (4) under transformations (5) were studied in [10], where four independent fourth-order invariants of Eq. (4) were constructed. However, the approach used there, which is based on the evaluation of pseudovector fields, does not permit one to prove that these invariants form a basis. By comparing the dimension of the space of variables on which an invariant depends with the number of conditions from which it is found [just as in the case of Eq. (1)], we see that Eq. (4) has six fourth-order basis invariants.

Note also that the problem of finding a transformation (2) relating two equations of the form (4) was solved by Svinolupov by bringing the system of equations for the functions $\varphi(x)$ and $\psi(x,y)$ into involution.

2. INVARIANTS OF EQ. (1)

An invariant I of the group E of equivalence transformations of Eq. (1) is found from the equation [5, Sec. 6.4] $XI = 0$, where

$$\begin{aligned} X = {} & \xi(x,y,P,Q,R)\partial_x + \eta(x,y,P,Q,R)\partial_y + \pi(x,y,P,Q,R)\partial_P \\ & + \varkappa(x,y,P,Q,R)\partial_Q + \varrho(x,y,P,Q,R)\partial_R \end{aligned} \tag{6}$$

is the differential operator corresponding to the group E. To find the coordinates of the operator X, we compute its continuation to the derivatives y' and y'':

$$\tilde{X} = X + (D\eta - y'D\xi)\,\partial_{y'} + \left(D^2\eta - y'D^2\xi - 2y''D\xi\right)\partial_{y''},$$

where $D = \partial_x + y'\partial_y + y''\partial_{y'} + (P_x + y'P_y)\,\partial_P + (P_{xx} + y'P_{xy})\,\partial_{P_x} + \cdots$ A criterion for the invariance of Eq. (1) under X has the form

$$\tilde{X}\left(y'' - Ry'^2 - 2Qy' - P\right)\Big|_{y''=Ry'^2+2Qy'+P} = 0.$$

Matching the coefficients of like powers of y' and the derivatives of the functions P, Q, and R in this relation gives the system of determining equations

$$\begin{gathered} \xi_P = 0, \qquad \xi_Q = 0, \qquad \xi_R = 0, \qquad \eta_P = 0, \qquad \eta_Q = 0, \qquad \eta_R = 0, \\ \pi = \eta_{xx} + P(\eta_y - 2\xi_x) - 2Q\eta_x, \qquad 2\varkappa = 2\eta_{xy} - \xi_{xx} - 3P\xi_y - 2Q\xi_x - 2R\eta_x, \\ \varrho = \eta_{yy} - 2\xi_{xy} - 4Q\xi_y - R\eta_y, \qquad \xi_{yy} + R\xi_y = 0 \end{gathered} \tag{7}$$

for the coordinates of the operator (6). Since $\xi_R = 0$, it follows from the last equation in (7) that $\xi_y = 0$; from the other equations, one can find π, $\varkappa$, and ϱ. Therefore, the group E of equivalence transformations of Eq. (1) is generated by the operator

$$\begin{aligned} X = {} & \xi(x)\partial_x + \eta(x,y)\partial_y + \left(\eta_{xx} + P(\eta_y - 2\xi') - 2Q\eta_x\right)\partial_P \\ & + \left(\eta_{xy} - \xi''/2 - Q\xi' - R\eta_x\right)\partial_Q + \left(\eta_{yy} - R\eta_y\right)\partial_R. \end{aligned} \tag{8}$$

Since the functions $\xi(x)$ and $\eta(x,y)$ and all of their derivatives in (8) are arbitrary, it follows from the theory of invariants of infinite transformation groups [5, p. 326] that the relation $XI = 0$

should be split with respect to these functions. Then the operator X splits into 8 operators

$$\begin{gathered} X_1 \equiv X(\xi) = \partial_x, \qquad X_2 \equiv X(\eta) = \partial_y, \qquad X_3 \equiv X(\eta_{xx}) = \partial_P, \\ X_4 \equiv X(\eta_{xy}) = -2X(\xi'') = \partial_Q, \qquad X_5 \equiv X(\eta_{yy}) = \partial_R, \\ X_6 \equiv X(\eta_x) = -2Q\partial_P - R\partial_Q, \qquad X_7 \equiv X(\eta_y) = P\partial_P - R\partial_R, \\ X_8 \equiv X(\xi') = -2P\partial_P - Q\partial_Q, \end{gathered}$$

which form a Lie algebra L_8, and the invariant $I(x, y, P, Q, R)$ is a solution of the system $X_i I = 0$, $i = 1, \dots, 8$. This system has only the trivial solution $I = \text{const}$; consequently, the invariants of Eq. (1) are differential. The continuation of the operator (8) to the derivatives of the functions P, Q, and R can be computed by standard formulas [5, Sec. 4.8], where x and y play the role of independent variables. The comparison of the dimension of the space in which the continued operator $\underset{(k)}{X}$, $k = 1, 2, \dots$, acts with the number of the derivatives of the functions ξ and η on which its coordinates depend shows that Eq. (1) has nontrivial invariants of order ≥ 3.

By matching the coefficients of ξ, η, ξ', η_x, η_y, $\dots$, η_{yyyyy} in the operator $\underset{(3)}{X}$, we obtain operators $X_1, \dots, X_{27}$ that act in the space $R^{32}(x, y, P, Q, R, P_x, P_y, \dots, R_{yyy})$ and form a Lie algebra L_{27}. A third-order invariant of Eq. (1) is a solution of the system $X_i I = 0$, $i = 1, \dots, 27$. A basis of functionally independent solutions of the subsystem $X_i I = 0$, $i = 1, \dots, 15$, where

$$\begin{aligned} &X_1 \equiv X(\xi) = \partial_x, \qquad X_2 \equiv X(\eta) = \partial_y, \qquad X_3 \equiv X(\eta_{xxxxx}) = \partial_{P_{xxx}}, \\ &X_4 \equiv X(\eta_{xxxxy}) = \partial_{P_{xxy}} + \partial_{Q_{xxx}}, \qquad X_5 \equiv X(\eta_{xxxyy}) = \partial_{P_{xyy}} + \partial_{Q_{xxy}} + \partial_{R_{xxx}}, \\ &X_6 \equiv X(\eta_{xxyyy}) = \partial_{P_{yyy}} + \partial_{Q_{xyy}} + \partial_{R_{xxy}}, \qquad X_7 \equiv X\left(\xi^V\right) = -\frac{1}{2}\partial_{Q_{xxx}}, \\ &X_8 \equiv X(\eta_{xyyyy}) = \partial_{Q_{yyy}} + \partial_{R_{xyy}}, \qquad X_9 \equiv X(\eta_{yyyyy}) = \partial_{R_{yyy}}, \\ &X_{10} \equiv X(\eta_{xxxx}) = \partial_{P_{xx}} - 2Q\partial_{P_{xxx}} - R\partial_{Q_{xxx}}, \\ &X_{11} \equiv X\left(\xi^{IV}\right) = -\frac{1}{2}\partial_{Q_{xx}} - 2P\partial_{P_{xxx}} - Q\partial_{Q_{xxx}}, \\ &X_{12} \equiv X(\eta_{xxxy}) = \partial_{P_{xy}} + \partial_{Q_{xx}} + P\partial_{P_{xxx}} - 2Q\partial_{P_{xxy}} - R\left(\partial_{Q_{xxy}} + \partial_{R_{xxx}}\right), \\ &X_{13} \equiv X(\eta_{xxyy}) = \partial_{P_{yy}} + \partial_{Q_{xy}} + \partial_{R_{xx}} + P\partial_{P_{xxy}} - 2Q\partial_{P_{xyy}} - R\left(\partial_{Q_{xyy}} + \partial_{R_{xxy}}\right), \\ &X_{14} \equiv X(\eta_{xyyy}) = \partial_{Q_{yy}} + \partial_{R_{xy}} + P\partial_{P_{xyy}} - 2Q\partial_{P_{yyy}} - R\left(\partial_{Q_{yyy}} + \partial_{R_{xyy}}\right), \\ &X_{15} \equiv X(\eta_{yyyy}) = \partial_{R_{yy}} + P\partial_{P_{yyy}} - R\partial_{R_{yyy}}, \end{aligned}$$

is given by the functions

$$\begin{gathered} P, \quad Q, \quad R, \quad P_x, \quad P_y, \quad Q_x, \quad Q_y, \quad R_x, \quad R_y, \\ w_1 = R_{xx} - Q_{xy}, \qquad w_2 = R_{xy} - Q_{yy}, \qquad w_3 = R_{xx} - 2Q_{xy} + P_{yy}, \\ t_1 = w_{1x}, \qquad t_2 = w_{1y}, \qquad t_3 = w_{2y}, \qquad t_4 = w_{3x} - RP_{xy} + 2QQ_{xy} - PR_{xy}, \\ t_5 = w_{3y} - RP_{yy} + 2QQ_{yy} - PR_{yy}. \end{gathered} \tag{9}$$

In the variables (9), the next five operators of the algebra L_{27} acquire the form

$$\begin{aligned} &X_{16} \equiv X(\eta_{xxx}) = \partial_{P_x} + R_y\partial_{t_4}, \\ &X_{17} \equiv X(\xi''') = -\frac{1}{2}\partial_{Q_x} + (Q_y - R_x)\,\partial_{t_1} + (2Q_y - R_x)\,\partial_{t_4}, \\ &X_{18} \equiv X(\eta_{xxy}) = \partial_{P_y} + \partial_{Q_x} + R\partial_{w_3} + (Q_y - R_x)\,\partial_{t_1} + (R_x - 2Q_y)\,\partial_{t_4} + 2R_y\partial_{t_5}, \\ &X_{19} \equiv X(\eta_{xyy}) = \partial_{Q_y} + \partial_{R_x} - 2Q\partial_{w_3} + (Q_y - R_x)\,\partial_{t_2} + (P_y - 2Q_x)\,\partial_{t_4} - 4Q_y\partial_{t_5}, \\ &X_{20} \equiv X(\eta_{yyy}) = \partial_{R_y} + P\partial_{w_3} + (Q_y - R_x)\,\partial_{t_3} + P_x\partial_{t_4} + 2P_y\partial_{t_5}. \end{aligned}$$

For invariants of the operators $X_{16}, \ldots, X_{20}$, one can take the functions

$$\begin{gathered}P, \quad Q, \quad R, \quad M=R_x-Q_y,\\ v_1=w_1+QM, \qquad v_2=w_2+RM, \qquad v_3=w_3-RP_y+2QQ_y-PR_y,\\ \tau_1=M_{xx}+3(PM)_y-2(QM)_x, \qquad \tau_2=M_{xy}+2(QM)_y-(RM)_x,\\ \tau_3=M_{yy}+(RM)_y, \quad \tau_4=v_{3x}+2(PM)_y-2(QM)_x, \qquad \tau_5=v_{3y}+2(QM)_y-2(RM)_x;\end{gathered} \tag{10}$$

then in the variables (10), we have

$$\begin{aligned}X_{21} &\equiv X\left(\eta_{xx}\right)=\partial_P+2v_2\left(\partial_{\tau_1}+\partial_{\tau_4}\right),\\ X_{22} &\equiv X\left(\eta_{xy}\right)=\partial_Q-4v_1\partial_{\tau_1}-\left(2v_1+v_3\right)\partial_{\tau_4}+2v_2\partial_{\tau_5},\\ X_{23} &\equiv X\left(\eta_{yy}\right)=\partial_R-2v_1\partial_{\tau_2}-2v_2\partial_{\tau_3}-\left(2v_1+v_3\right)\partial_{\tau_5}.\end{aligned}$$

In terms of the invariants

$$\begin{gathered}M, \quad v_1, \quad v_2, \quad v_3,\\ k_1=\tau_1-2Pv_2+4Qv_1, \qquad k_2=\tau_2+2Rv_1, \qquad k_3=\tau_3+2Rv_2,\\ k_4=\tau_4-2Pv_2+Q\left(2v_1+v_3\right), \qquad k_5=\tau_5-2Qv_2+R\left(2v_1+v_3\right)\end{gathered}$$

of the operators X_{21}, X_{22}, and X_{23}, the remaining operators of the algebra L_{27} acquire the form

$$\begin{aligned}X_{24} &\equiv X\left(\xi''\right)=-M\left(\frac{3}{2}\partial_{v_1}+\partial_{v_3}\right)-4v_1\partial_{k_1}-2v_2\partial_{k_2}-\left(v_1+\frac{5}{2}v_3\right)\partial_{k_4}-v_2\partial_{k_5},\\ X_{25} &\equiv X\left(\eta_x\right)=-v_2\partial_{v_1}-2k_2\partial_{k_1}-k_3\partial_{k_2}-k_5\partial_{k_4},\\ X_{26} &\equiv X\left(\eta_y\right)=-M\partial_M-v_1\partial_{v_1}-2v_2\partial_{v_2}-v_3\partial_{v_3}-k_1\partial_{k_1}-2k_2\partial_{k_2}\\ &\quad -3k_3\partial_{k_3}-k_4\partial_{k_4}-2k_5\partial_{k_5},\\ X_{27} &\equiv X\left(\xi'\right)=-M\partial_M-2v_1\partial_{v_1}-v_2\partial_{v_2}-2v_3\partial_{v_3}-3k_1\partial_{k_1}-2k_2\partial_{k_2}\\ &\quad -k_3\partial_{k_3}-3k_4\partial_{k_4}-2k_5\partial_{k_5}.\end{aligned}$$

The invariants of the operators $X_{24}, \ldots, X_{27}$ are the functions

$$I_1=\frac{L_1}{M^6}, \qquad I_2=\frac{L_2}{M^3 v_2}, \qquad I_3=\frac{L_3}{v_2^2}, \qquad I_4=\frac{v_2 L_4}{M^6}, \qquad I_5=\frac{L_5}{M^3}; \tag{11}$$

here we have used the notation

$$\begin{gathered}V=\frac{3}{2}v_3-v_1, \qquad l_1=v_2^2k_1+2v_2Vk_2+V^2k_3, \qquad l_2=v_2k_2+Vk_3,\\ l_4=v_2k_4+Vk_5, \qquad L_1=Ml_1-3v_2^2v_3^2, \qquad L_2=Ml_2-2v_2^2v_3,\\ L_3=Mk_3, \qquad L_4=Ml_4-2v_2v_3^2, \qquad L_5=Mk_5-v_2v_3.\end{gathered} \tag{12}$$

In addition, one can readily see that the operators $X_{24}, \ldots, X_{27}$ leave invariant the separate equations

$$M=0; \qquad v_2=0; \qquad k_3=0 \tag{13}$$

and systems

$$M=0, \quad v_3=0; \qquad v_2=0, \quad V=0; \qquad v_2=0, \quad k_5=0. \tag{14}$$

The condition for the invariance of Eqs. (13) and systems (14) under the remaining operators in L_{27} is identically valid. Consequently, they are invariant under the general operator (8) of equivalence transformations of Eq. (1).

The functions (11) form a basis of functionally independent solutions of the system $X_i I=0$, $i=1,\ldots,27$. Other invariants of Eq. (1) can be derived from (11) by application of the invariant differentiation operators $\mathscr{D}=fD_x+gD_y$ with coefficients f and g depending on x, y, the functions

P, Q, and R, and their derivatives with respect to x and y. They are found from the system of equations [5, Sec. 24.2]

$$\underset{(3)}{X} f = f\xi_x + g\xi_y, \qquad \underset{(3)}{X} g = f\eta_x + g\eta_y. \tag{15}$$

By matching the coefficients of ξ and η and their derivatives in (15), for the functions f and g, we obtain the system of equations $X_i f = 0$, $X_i g = 0$, $i = 1, \ldots, 24$, $X_{25} f = 0$, $X_{26} f = 0$, $X_{27} f = f$, $X_{25} g = f$, $X_{26} g = g$, $X_{27} g = 0$. Its solution has the form $f = Fv_2/M^2$, $g = FV/M^2 + GM/v_2$, where F and G are arbitrary functions of the invariants (11). This provides two independent operators

$$\mathscr{D}_1 = \frac{v_2}{M^2} D_x + \frac{V}{M^2} D_y, \qquad \mathscr{D}_2 = \frac{M}{v_2} D_y. \tag{16}$$

The result obtained can be stated in the form of the following assertion.

Theorem 1. *A basis of invariants of Eq.* (1) *is formed by the five third-order invariants* (11).

Proof. Let us show that the invariants (11) form a basis. It follows from (10) that the third-order derivatives of the functions P, Q, and R occur in (11) via the quantities M_{xx}, M_{xy}, M_{yy}, v_{3x}, and v_{3y}. The application of the operators (16) to the invariants (11) provides 7 independent fourth-order invariants depending on M_{xxx}, M_{xxy}, M_{xyy}, M_{yyy}, v_{3xx}, v_{3xy}, and v_{3yy}. The continued operator $\underset{(4)}{X}$ acts in the space $R^{47}(x, y, P, Q, R, P_x, P_y, \ldots, R_{yyyy})$. The splitting of the equation $\underset{(4)}{X} I = 0$ with respect to ξ, η, ξ', η_x, η_y, $\ldots$, η_{yyyyyy} provides a system of 35 equations. Its solution is formed by 12 independent invariants, 5 of which are the invariants (11). For the remaining 7 invariants one can take the fourth-order invariants obtained from (11) by application of the invariant differentiation operators (16).

By repeating these considerations, one can show that, at any step n, the number

$$(n + 3) + (n + 2)$$

of independent invariants of order $n + 3$ obtained after the n-multiple differentiation of the invariants (11) is equal to the difference of the number $3(n + 4)$ of $(n + 3)$rd-order derivatives of the functions P, Q, and R and the number $1 + (n + 6)$ of $(n + 5)$th-order derivatives of the functions $\xi(x)$ and $\eta(x, y)$, which completes the proof.

3. INVARIANTS OF EQ. (1) IN DEGENERATE CASES

If the coefficients of Eq. (1) satisfy the condition $R_x = Q_y$ (respectively, $M = 0$) or

$$R_{xy} - Q_{yy} + R\,(R_x - Q_y) = 0$$

(respectively, $v_2 = 0$), then it is impossible to compute the invariants (11). For example, $M = 0$ for all six Painlevé equations. Below, for the model case in which $M = 0$, we show that if Eq. (1) is supplemented with a condition that is invariant under the operator (8), then the coordinates of the admitted operator of equivalence transformations remain the same.

To this end, we compute the continuation of the operator (6) to the derivatives y', y'', Q_y, and R_x :

$$\hat{X} = \tilde{X} + \left(\bar{D}_y \varkappa - Q_x \bar{D}_y \xi - Q_y \bar{D}_y \eta\right) \partial_{Q_y} + \left(\bar{D}_x \varrho - R_x \bar{D}_x \xi - R_y \bar{D}_x \eta\right) \partial_{R_x},$$

where

$$\bar{D}_x = \partial_x + P_x \partial_P + Q_x \partial_Q + R_x \partial_R + \cdots, \qquad \bar{D}_y = \partial_y + P_y \partial_P + Q_y \partial_Q + R_y \partial_R + \cdots$$

A criterion for the invariance of Eq. (1) and the equation $M = 0$ under the operator (6) has the form

$$\begin{aligned} \hat{X}\,(R_x - Q_y)\Big|_{R_x = Q_y} &= 0, \\ \tilde{X}\left(y'' - Ry'^2 - 2Qy' - P\right)\Big|_{y'' = Ry'^2 + 2Qy' + P,\ R_x = Q_y,\ R_{xx} = Q_{xy},\ R_{xy} = Q_{yy}} &= 0. \end{aligned} \tag{17}$$

Matching the coefficients of like powers of P_x, P_y, Q_x, Q_y, and R_y in the first relation in (17) gives the determining equations

$$\begin{gathered}\xi_P = 0, \qquad \eta_P = 0, \qquad \xi_R - \eta_Q = 0, \qquad \varrho_P = 0, \qquad \varkappa_P = 0,\\ \varrho_Q + \xi_y = 0, \qquad \varkappa_R + \eta_x = 0, \qquad \varrho_R - \varkappa_Q + \eta_y - \xi_x = 0.\end{gathered} \tag{18}$$

The coefficients of the powers of Q_{xx}, R_{yy}, and y' in the second equation (17) provide the equations

$$\xi_Q = 0, \qquad \eta_Q = 0, \qquad \eta_R = 0;$$

i.e., $\xi = \xi(x, y)$, $\eta = \eta(x, y)$, and the remaining determining equations coincide with (7). Consequently, $\xi_y = 0$, and condition (18) is satisfied for π, $\varkappa$, and ϱ found from (7). Therefore, the simultaneous analysis of Eq. (1) and the equation $M = 0$ leads to the same operator (8) and does not permit one to find invariants of Eq. (1) for the case in which $R_x = Q_y$, since the equation $XI = 0$ has the same solution (11).

For the case in which $M = 0$, one can compute the invariants by representing Eq. (1) in the form

$$y'' = S_y(x, y)y'^2 + 2S_x(x, y)y' + P(x, y). \tag{19}$$

The operator of equivalence transformations of Eq. (19) can be computed by analogy with (8) and has the form

$$\begin{aligned}X = \xi(x)\partial_x + \eta(x, y)\partial_y + (\eta_{xx} + P(\eta_y - 2\xi') - 2S_x\eta_x)\,\partial_P + (K + \eta_y - \xi'/2)\,\partial_S\\ + (\eta_{xy} - \xi''/2 - S_x\xi' - S_y\eta_x)\,\partial_{S_x} + (\eta_{yy} - S_y\eta_y)\,\partial_{S_y}, \quad K = \text{const}.\end{aligned} \tag{20}$$

One can readily see that the change of variables $S_x = Q$, $S_y = R$ in the coordinates of the operator (20) in ∂_P, ∂_{S_x}, and ∂_{S_y} reduces it to an operator coinciding with (8). A basis of invariants of Eq. (19) is found from the equation $XI = 0$ with the operator (20). By performing the change of variables $S_x = Q$, $S_y = R$, $S_{xx} = Q_x$, $S_{xy} = Q_y$, $S_{yy} = R_y$, $\ldots$ in these invariants, one can obtain invariants of Eq. (1) for the case in which $M = 0$.

The same result can be obtained without passing from Eq. (1) to Eq. (19) if one directly computes the invariants of Eq. (1) on the manifold $M = 0$. To this end, in the coordinates of the continued operator $\underset{(3)}{X}$ given by (8), we make the substitution

$$\begin{gathered}R_x = Q_y, \qquad R_{xx} = Q_{xy}, \qquad R_{xy} = Q_{yy},\\ R_{xxx} = Q_{xxy}, \qquad R_{xxy} = Q_{xyy}, \qquad R_{xyy} = Q_{yyy},\end{gathered} \tag{21}$$

and the action of the operator $\underset{(3)}{X}$ is accordingly restricted to the variables

$$\begin{gathered}x, \quad y, \quad P, \quad Q, \quad R, \quad P_x, \quad P_y, \quad Q_x, \quad Q_y, \quad R_y,\\ P_{xx}, \quad \ldots, \quad Q_{yy}, \quad R_{yy}, \quad P_{xxx}, \quad \ldots, \quad Q_{yyy}, \quad R_{yyy}.\end{gathered} \tag{22}$$

By matching the coefficients of $\xi, \eta, \xi', \eta_x, \eta_y, \ldots, \eta_{yyyyy}$ in $\underset{(3)}{X}$, we obtain operators $X_1, \ldots, X_{27}$ acting in the space R^{26} of the variables (22). Consequently, if $M = 0$, then Eq. (1) has no third-order invariants, since the system $X_iI = 0$, $i = 1, \ldots, 27$, has only the trivial solution $I = \text{const}$.

By computing the fourth continuation of the operator (8) to the variables

$$P_{xxxx}, \quad \ldots, \quad P_{yyyy}, \quad Q_{xxxx}, \quad \ldots, \quad Q_{yyyy}, \quad R_{yyyy}, \tag{23}$$

by making the substitution (21) and $R_{xxxx} = Q_{xxxy}$, $R_{xxxy} = Q_{xxyy}$, $R_{xxyy} = Q_{xyyy}$, $R_{xyyy} = Q_{yyyy}$ in its coordinates, and by splitting $\underset{(4)}{X}$ with respect to $\xi, \eta, \xi', \eta_x, \eta_y, \ldots, \eta_{yyyyyy}$, one can obtain the operators $X_1, \ldots, X_{35}$ acting in the space R^{37} of the variables (22) and (23). Therefore, if $M = 0$, then Eq. (1) has two fourth-order invariants. The results of computation of invariants of Eq. (1) in the main and degenerate cases are summarized in the following assertion.

Theorem 2. *All equations of the class* (1) *belong to one of five types. Depending on it, their basis invariants and invariant differentiation operators have the following form.*

1. ($M \neq 0$, $v_2 \neq 0$) *The basis invariants and the operators* $\mathscr{D}_1$ *and* $\mathscr{D}_2$ *are given by* (11) *and* (16).
2. ($M \neq 0$, $v_2 = 0$, $V \neq 0$)

$$I_1 = \frac{k_5}{M^2}, \qquad I_2 = \frac{K_5}{(MV)^2}; \qquad \mathscr{D}_1 = \frac{M}{V} D_x + \frac{K_1}{4M^2 V} D_y, \qquad \mathscr{D}_2 = \frac{V}{M^2} D_y.$$

3. ($M = 0$, $v_3 \neq 0$, $k_5 \neq 0$)

$$I_1 = \frac{v_3 \lambda_3}{k_5^2}, \qquad I_2 = \frac{\Lambda_3}{v_3^3 k_5};$$
$$\mathscr{D}_1 = \frac{5v_3\lambda_3 - 6k_5^2}{v_3 k_5^{3/2}} D_x + \frac{6k_4k_5 - 5v_3\lambda_2}{v_3 k_5^{3/2}} D_y, \qquad \mathscr{D}_2 = \frac{v_3}{k_5} D_y.$$

4. ($M = 0$, $v_3 \neq 0$, $k_5 = 0$)

$$I_1 = \frac{t_3}{t_1^2}; \qquad \mathscr{D}_1 = \frac{v_3}{t_1^{1/2}} D_x - \frac{t_2}{5v_3^3 t_1^{1/2}} D_y, \qquad \mathscr{D}_2 = \frac{t_1}{v_3^2} D_y.$$

5. ($v_2 = 0$, $V = 0$) *Equation* (1) *can be linearized by a point transformation.*

The computation uses the auxiliary quantities (12) and

$$M = R_x - Q_y, \qquad N = P_y - Q_x + Q^2 - PR,$$
$$v_1 = M_x + QM, \qquad v_2 = M_y + RM, \qquad v_3 = M_x + N_y, \qquad V = \frac{3}{2}v_3 - v_1,$$
$$k_1 = v_{1x} + Pv_2 + Qv_1 + 3MN, \qquad k_2 = v_{2x} + 2Qv_2 - 2M^2, \qquad k_3 = v_{2y} + 2Rv_2,$$
$$k_4 = v_{3x} + Qv_3 + 2MN, \qquad k_5 = v_{3y} + Rv_3 - 2M^2;$$
$$K_1 = Mk_1 - \frac{4}{3}v_1^2, \qquad K_4 = Mk_4 - v_3\left(v_1 + \frac{1}{2}v_3\right), \qquad K_5 = k_5K_1 + 4M^2K_4;$$
$$\lambda_1 = k_{4x} + Pk_5 + Qk_4 + 5Nv_3, \qquad \lambda_2 = k_{5x} + 2Qk_5, \qquad \lambda_3 = k_{5y} + 2Rk_5,$$
$$\Lambda_1 = \lambda_2^2 - \lambda_1\lambda_3, \qquad \Lambda_2 = k_5^2\lambda_1 - 2k_4k_5\lambda_2 + k_4^2\lambda_3, \qquad \Lambda_3 = 5v_3\Lambda_1 + 6\Lambda_2;$$
$$s_1 = \lambda_{1x} + Q\lambda_1 + 12Nk_4, \qquad s_2 = s_{1x} + Qs_1 + 21N\lambda_1 + 5Pv_3^2,$$
$$t_1 = v_3\lambda_1 - \frac{6}{5}k_4^2, \qquad t_2 = v_3^2 s_1 - \frac{21}{5}k_4\left(v_3\lambda_1 - \frac{4}{5}k_4^2\right),$$
$$t_3 = v_3^3 s_2 - \frac{32}{5}v_3^2 k_4 s_1 + \frac{336}{25}k_4^2\left(v_3\lambda_1 - \frac{3}{5}k_4^2\right).$$

The assertion of Theorem 2 in case 1 follows from Theorem 1, and in the cases 2, 3, and 4, the proof is similar to that of Theorem 1. In case 5, the coefficients of Eq. (1) satisfy the assumptions of the theorem on the linearization of a second-order ordinary differential equation [9] (see also [11]). The substitution (5) [(2) if $M = 0$] linearizing Eq. (1) can be found by the method in [11, Sec. 2.2]. An invariant of the linear equation $y'' + 2q(x)y' + p(x)y = 0$ under the transformation $\bar{y} = \sigma(x)y$ is given by the coefficient $J(x) = p - q^2 - q'$ of this equation reduced to the normal form [12, Sec. 25.1] $\bar{y}'' + J(x)\bar{y} = 0$.

4. EXAMPLES

In addition to six Painlevé equations, reduction of integrable equations of mathematical physics often leads to the ordinary differential equation [1, 2]

$$y'' = \frac{y'^2}{2y} - \frac{1}{2y} - \bar{x}y + 4\alpha y^2, \qquad \alpha = \text{const.} \tag{24}$$

Its solution $y(\bar{x})$ is related by the transformation $2\alpha y = w' + w^2 + \bar{x}/2$ to the solution $w(\bar{x})$ of the second Painlevé equation $w'' = 2w^3 + \bar{x}w - 2\alpha - 1/2$. Equation (24) is an equation of the third type, and its basis of invariants is formed by the functions

$$
\begin{aligned}
J_1 &= -\frac{15}{2\alpha y^3}\left(1 - \frac{1}{4\alpha y^3}\right)\left(1 + \frac{5}{4\alpha y^3}\right)^{-2}, \\
J_2 &= 16\left(1 + \frac{35}{4\alpha y^3}\right)\left(1 - \frac{3}{64\alpha y^3} - \frac{5}{128\alpha^2 y^6} - \frac{3\bar{x}}{32\alpha y}\right)\left(1 - \frac{1}{4\alpha y^3}\right)^{-3}\left(1 + \frac{5}{4\alpha y^3}\right)^{-1}.
\end{aligned}
\tag{25}
$$

The use of the invariant differentiation operators

$$
\begin{aligned}
\mathscr{D}_1 &= -\left(\frac{3}{\alpha y}\right)^{1/2}\left(1 + \frac{35}{4\alpha y^3}\right)\left(1 - \frac{1}{4\alpha y^3}\right)^{-1}\left(1 + \frac{5}{4\alpha y^3}\right)^{-3/2} D_{\bar{x}}, \\
\mathscr{D}_2 &= 2y\left(1 - \frac{1}{4\alpha y^3}\right)\left(1 + \frac{5}{4\alpha y^3}\right)^{-1} D_y
\end{aligned}
$$

permits one to derive other invariants of Eq. (24) from the functions (25); for example,

$$
J_3 = \mathscr{D}_1 J_2 = \frac{1}{2}\left(\frac{3}{\alpha y}\right)^{3/2}\left(1 + \frac{35}{4\alpha y^3}\right)^2\left(1 - \frac{1}{4\alpha y^3}\right)^{-4}\left(1 + \frac{5}{4\alpha y^3}\right)^{-5/2}.
$$

Consider the equations of second approximation in shallow water theory:

$$
h_t + (uh)_x = 0, \qquad u_t + uu_x + gh_x = \frac{1}{3h}\left(h^3\left(u_{xt} + uu_{xx} - u_x^2\right)\right)_x, \qquad g = \text{const}
$$

(the Green–Naghdi model [13, 14]). The reduction with respect to the sum of the time translation and Galilei transformation, $u = t + v(z)$, $h = h(z)$, $z = x - t^2/2$, leads to the system of ordinary differential equations [15]

$$
(hv)' = 0, \qquad 1 + vv' + gh' = \frac{1}{3h}\left(h^3\left(vv'' - v'^2\right)\right)',
$$

which can be twice integrated: $h = c_1 v^{-1}$,

$$
v'' = \frac{3}{2}\frac{v'^2}{v} + \frac{3}{c_1^2}\left(c_1 g + (z + c_2)v + v^3/2\right), \qquad c_1, c_2 = \text{const}. \tag{26}
$$

Equation (26) is an equation of the third type, and a basis of its invariants is formed by the functions

$$
\begin{aligned}
I_1 &= \frac{30v^3}{c_1 g}\left(1 + \frac{v^3}{c_1 g}\right)\left(1 - \frac{5v^3}{c_1 g}\right)^{-2}, \\
I_2 &= 16\left(1 - \frac{35v^3}{c_1 g}\right)\left(1 + \frac{3v^3}{16c_1 g} - \frac{5v^6}{8c_1^2 g^2} + \frac{3v(z + c_2)}{8c_1 g}\right)\left(1 + \frac{v^3}{c_1 g}\right)^{-3}\left(1 - \frac{5v^3}{c_1 g}\right)^{-1};
\end{aligned}
\tag{27}
$$

other invariants can be obtained from (27) with the use of the invariant differentiation operators

$$
\begin{aligned}
\mathscr{D}_1 &= -2\left(-\frac{c_1}{g}v\right)^{1/2}\left(1 - \frac{35v^3}{c_1 g}\right)\left(1 + \frac{v^3}{c_1 g}\right)^{-1}\left(1 - \frac{5v^3}{c_1 g}\right)^{-3/2} D_z, \\
\mathscr{D}_2 &= -2v\left(1 + \frac{v^3}{c_1 g}\right)\left(1 - \frac{5v^3}{c_1 g}\right)^{-1} D_v.
\end{aligned}
$$

The comparison of the expressions (25) and (27) shows that the equality $J_1 = I_1$, $J_2 = I_2$ of basic invariants of Eqs. (24) and (26) holds if

$$z + c_2 = \left(\frac{c_1 g}{4\alpha}\right)^{2/3} \bar{x}, \qquad v = -\left(\frac{c_1 g}{4\alpha}\right)^{1/3} y^{-1}. \tag{28}$$

The requirement of the equality of other invariants of these equations, in particular, J_3 and

$$I_3 = \mathscr{D}_1 I_2 = -\frac{12v}{c_1 g}\left(-\frac{c_1 v}{g}\right)^{1/2}\left(1 - \frac{35v^3}{c_1 g}\right)^2\left(1 + \frac{v^3}{c_1 g}\right)^{-4}\left(1 - \frac{5v^3}{c_1 g}\right)^{-5/2},$$

imposes the constraint $4\alpha = \sqrt{3}g$ on the parameters of Eqs. (24) and (26). One can obtain the same relationship between the parameters α and g by substituting (28) into (26) and by comparing the resulting relation with Eq. (24). Consequently, the change of variables $z + c_2 = (c_1^2/3)^{1/3}\,\bar{x}$, $v = -\left(c_1^2/3\right)^{1/6} y^{-1}$ reduces Eq. (26) to the form (24) with $\alpha = \sqrt{3}g/4$.

As another example, we consider the nonlinear Schrödinger equation with potential,

$$iu_t + u_{xx} + |u|^2 u - V(x)u = 0, \qquad V(x) \neq \text{const},$$

and find functions $V(x)$ for which its stationary solution $u = \varrho(x)\exp(i\varphi(x))$ can be found by integration of Eq. (24). The functions $\varrho(x)$ and $\varphi(x)$ satisfy the system of ordinary differential equations

$$\varrho\varphi'' + 2\varrho'\varphi' = 0, \qquad \varrho'' + \varrho^3 - \varrho\varphi'^2 - V(x)\varrho = 0.$$

By expressing $\varphi' = k/\varrho^2$, $k = \text{const}$, from the first equation of the system and by substituting it into the second equation, for the function $\varrho(x)$, we obtain the third-type equation

$$\varrho'' = V(x)\varrho - \varrho^3 + k^2/\varrho^3. \tag{29}$$

Its invariants are

$$\begin{aligned} I_1 &= -\frac{60k^2}{\varrho^6}\left(1 - \frac{2k^2}{\varrho^6}\right)\left(1 + \frac{10k^2}{\varrho^6}\right)^{-2}, \\ I_2 &= 16\left(1 + \frac{70k^2}{\varrho^6}\right)\left(1 - \frac{3k^2}{8\varrho^6} - \frac{5k^4}{2\varrho^{12}} - \frac{3V(x)}{8\varrho^2}\right)\left(1 - \frac{2k^2}{\varrho^6}\right)^{-3}\left(1 + \frac{10k^2}{\varrho^6}\right)^{-1}. \end{aligned} \tag{30}$$

The condition of equality of the invariants (25) and (30) implies the relations

$$\bar{x} = 2(\alpha/k)^{2/3} V(x), \qquad y = \left(\alpha k^2\right)^{-1/3} \varrho^2/2. \tag{31}$$

By comparing higher-order invariants of Eqs. (24) and (29) or, simply, by substituting (31) into (24), one can obtain the constraints $V'' = 0$ and $4\alpha^2 V'^2 = -k^2$ for the parameters α and $V(x)$. Therefore, it is only in the case of a linear function $V(x) = V_1 x + V_0$, where V_0 and $V_1 = \text{const}$, that Eq. (29) is related to Eq. (24) [where $\alpha = (1/2)ik/V_1$] by the change of variables (31).

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