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BRIEF COMMUNICATIONS

Integrable Main Resonance Equations

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1. INTRODUCTION

The main resonance equation for a complex function $u(t)$,

$$iu' - \varphi(t)u + \lambda|u|^2u = F(t), \qquad F(t) = f(t)\,e^{i\theta(t)}, \tag{1}$$

arises when an ordinary differential equation with weak cubic nonlinearity is averaged over rapid one-frequency oscillations [1, Secs. 14, 18], [2, Chap. 9]. This equation is used to describe resonance phenomena in a number of problems of celestial mechanics [3], plasma physics [4], acceleration of relativistic particles [5], etc. Parametric resonance is described by the equation [1, Sec. 16]

$$iu' - \varphi(t)u + \lambda|u|^2u = F(t)\bar{u}, \qquad F(t) = f(t)\,e^{2i\theta(t)}\,. \tag{2}$$

We also consider the equation

$$iu' - \varphi(t)u + \lambda|u|^2u = \frac{i}{2}f(t)\left(e^{i\theta(t)} + e^{-i\theta(t)}\,\frac{u}{\bar{u}}\right) \tag{3}$$

describing the motion of particles in accelerators with constant controlling magnetic field when they approach the point at which phase stability fails [5, Secs. 4.7, 7.4], [6] and the equation

$$iu' - \varphi(t)u + \lambda|u|^2u = f(t)\,e^{i\theta(t)}\,|u| + ig(t)u, \tag{4}$$

arising in the modeling of magnetic reversal processes in ferromagnets.

In Eqs. (1)–(4), $\bar{u}$ denotes complex conjugation of the function u, $\lambda = \text{const} \neq 0$, $\lambda \in \mathbb{R}$, and $f(t)$, $\theta(t)$, $\varphi(t)$, and $g(t)$ are assumed smooth real functions. Using the Painlevé test [7], we study the integrability of the equations for the real function $\rho(t) = |u|^2$ which follow from (1)–(4). For the cases in which they are invariant with respect to translation and dilation, we construct the integrals of Eqs. (1)–(4).

2. INTEGRALS OF THE MAIN RESONANCE EQUATIONS

By substituting $u(t) = r(t)\,e^{i\psi(t)}$, we obtain an equivalent representation of Eqs. (1)–(4) in the form of the systems of equations

$$r' = -f(t)\sin(\psi - \theta(t)), \qquad \psi' = \lambda r^2 - \varphi(t) - \frac{f(t)}{r}\cos(\psi - \theta(t)), \tag{1$'$}$$

$$r' = -f(t)r\sin 2(\psi - \theta(t)), \qquad \psi' = \lambda r^2 - \varphi(t) - f(t)\cos 2(\psi - \theta(t)), \tag{2$'$}$$

$$r' = f(t)\cos(\psi - \theta(t)), \qquad \psi' = \lambda r^2 - \varphi(t), \tag{3$'$}$$

$$r' = -f(t)r\sin(\psi - \theta(t)) + g(t)r, \qquad \psi' = \lambda r^2 - \varphi(t) - f(t)\cos(\psi - \theta(t)) \tag{4$'$}$$

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for the real amplitude and phases functions. The elimination of the function $\psi(t)$ from (1′)–(4′) leads to equations of second order for the function $r(t)$. In what follows, however, it is more convenient to consider equations for the function $\rho(t) = r^2(t)$. For systems (1′)–(4′), they have, respectively, the form

$$\left(\rho'' - 2f^2 - \rho'\frac{f'}{f}\right)^2 = (\Phi - \lambda\rho)^2(4f^2\rho - \rho'^2), \tag{1''}$$

$$\left(\rho'' - 4f^2\rho - \rho'\frac{f'}{f}\right)^2 = 4(\Phi - \lambda\rho)^2(4f^2\rho^2 - \rho'^2), \tag{2''}$$

$$\left(\rho\rho'' - \frac{\rho'^2}{2} - \rho\rho'\frac{f'}{f}\right)^2 = \rho^2(\Phi - \lambda\rho)^2(4f^2\rho - \rho'^2), \tag{3''}$$

$$\begin{aligned}\left(\rho\rho'' - \frac{3}{2}\rho'^2 + \left(2g - \frac{f'}{f}\right)\rho\rho' + 2\rho^2\left(g\frac{f'}{f} - g' + f^2 - g^2\right)\right)^2 \\ = \rho^2(\Phi - \lambda\rho)^2(4f^2\rho^2 - (\rho' - 2g\rho)^2),,\end{aligned} \tag{4''}$$

where $\Phi(t) = \theta'(t) + \varphi(t)$. The invariance of Eqs. (1″)–(4″) with respect to point transformations allows us to find certain integrals of systems (1′)–(4′).

Proposition 1. 1. *If* $f(t) = C_1$, $\varphi(t) = C_2 - \theta'(t)$ (*here and elsewhere*, $C_i = \text{const}$), *then the integrals for systems* (1″)–(3″) *are the following relations*:

$$4C_1 r\cos(\psi - \theta(t)) + 2C_2 r^2 - \lambda r^4 = \text{const}$$

for system (1′);

$$2r^2(C_1\cos 2(\psi - \theta(t)) + C_2) - \lambda r^4 = \text{const},$$

for system (2′);

$$3C_1\sin(\psi - \theta(t)) + 3C_2 r - \lambda r^3 = \text{const}$$

for system (3′).

2. *If* $f(t) = C_1(t - t_0)^{-3/2}$, $\varphi(t) = C_2(t - t_0)^{-1} - \theta'(t)$, *then the integral for system* (1′) *is obtained by substituting*

$$R = r\sqrt{t - t_0}, \qquad \chi = \psi - \theta(t) \tag{5}$$

into the general solution of the equation

$$\frac{dR}{d\chi} = \frac{R^2/2 - C_1 R\sin\chi}{\lambda R^3 - C_2 R - C_1\cos\chi};$$

the integral for system (3′) *is obtained by substituting* (5) *into the general solution of the equation*

$$\frac{dR}{d\chi} = \frac{R/2 + C_1\cos\chi}{\lambda R^2 - C_2}.$$

If

$$f(t) = \frac{C_1}{t - t_0}, \qquad \varphi(t) = \frac{C_2}{t - t_0} - \theta'(t),$$

then the integral for system (2′) *is obtained by substituting* (5) *into the general solution of the equation*

$$\frac{dR}{d\chi} = \frac{R/2 - C_1 R\sin 2\chi}{\lambda R^2 - C_2 - C_1\cos 2\chi}.$$

3. *If*

$$g(t) = C_1 f(t) + \frac{f'(t)}{2f(t)}, \qquad \varphi(t) = C_2 f(t) - \theta'(t),$$

then the integral of system $(4')$ *is obtained from the general solution of the Abel equation of the second kind*

$$\frac{dR}{d\chi}=\frac{2R(C_1-\sin\chi)}{\lambda R-C_2-\cos\chi}$$

by substituting $R=r^2/f(t)$, $\chi=\psi-\theta(t)$. *In particular, if* $C_1=0$, *the integral is*

$$(3\cos(\psi-\theta(t))+C_2)rf^{-1/2}-\lambda r^3f^{-3/2}=\text{const}\,.$$

Proof. It is readily seen that, in case 1, Eqs. $(1'')$–$(3'')$ are invariant with respect to the symmetries of the translation $X_1=\partial_t$, and, in case 2, with respect to the symmetry of the dilation

$$X_2=(t-t_0)\partial_t-\rho\partial_\rho.$$

In case 3, Eq. $(4'')$ admits the operator

$$X_3=f(t)^{-1}\partial_t+f'(t)f(t)^{-2}\rho\partial_\rho.$$

They correspond to the symmetries

$$Z_1=\partial_t+\theta'(t)\partial_\psi \qquad \text{and} \qquad Z_2=2(t-t_0)(\partial_t+\theta'(t)\partial_\psi)-r\partial_r$$

of Eqs. $(1')$–$(3')$ and the symmetry

$$Z_3=2f(t)^{-1}(\partial_t+\theta'(t)\partial_\psi)+f'(t)f(t)^{-2}r\partial_r$$

of Eqs. $(4')$.

After the substitution of the invariants R, χ of the operators Z_1, Z_2, Z_3 ($R=r$ for Z_1) as new dependent variables, systems $(1')$–$(4')$ reduce to one equation for $R(\chi)$. Its general solution is given by the integral of Eqs. $(1')$–$(4')$ in the cases considered. □

3. INTEGRABILITY OF EQS. $(1'')$–$(4'')$

Applying the Painlevé test [7] and the weak Painlevé test [8] to Eqs. $(1'')$–$(4'')$, we obtain the following results.

Proposition 2. *Equations* $(3'')$, $(4'')$ *pass the weak Painlevé test*; *the expansion of the solution in a neighborhood of of the movable singular point* t_0 *is of the form*

$$\rho=(t-t_0)^{-1}\sum_{j=0}^{\infty}a_j(t-t_0)^{j/2}.$$

Proposition 3. *Equations* $(1'')$, $(2'')$ *pass the Painlevé test under the condition*

$$f(t)=C_1e^{C_0t}, \qquad \varphi(t)=C_2-\theta'(t). \tag{6}$$

If $C_0\neq 0$, *then the problem of integration of Eq.* (1) *can be reduced to the solution of the third Painlevé equation*

$$y''=\frac{y'^2}{y}-\frac{y'}{x}+\gamma y^3+\frac{1}{x}(\alpha y^2+\beta)+\frac{\delta}{y} \tag{7}$$

with parameters

$$\alpha=-\lambda C_1C_0^{-2}, \quad \beta=(C_2-iC_0)C_1C_0^{-2}, \quad \gamma=0, \quad \delta=C_1^2C_0^{-2},$$

and that of Eq. (2) *to the solution of the fifth Painlevé equation*

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{x} + \delta\frac{y(y+1)}{y-1} \tag{8}$$

with parameters

$$\alpha = 0, \quad \beta = (C_2/C_0 - i/2)^2/2, \quad \gamma = -C_1^2/(2C_0^2), \quad \delta = 0.$$

Proof. Substituting the expansion

$$\rho = (t-t_0)^{-p}\sum_{j=0}^{\infty} a_j(t-t_0)^j \tag{9}$$

of the solution in a neighborhood of the mobile singular point t_0 into $(1'')$, $(2'')$, we can determine the parameter $p = -1$ and the resonance $j = 2$, i.e., the power of the arbitrary constant appearing in (9). The coefficients a_j of the expansion (9) can be determined from a recurrence system whose first three equations are of the form

$$2m = -\lambda^2 a_0^2, \qquad m\frac{f_1}{f_0} = a_0(\lambda\Phi_0 - \lambda^2 a_1), \qquad 2m\frac{f_2}{f_0} - m\frac{f_1^2}{f_0^2} = \lambda a_0 \Phi_1. \tag{10}$$

Here $m = 2$ for Eq. $(1'')$, $m = 1/2$ for $(2'')$, and f_j, Φ_j denote the coefficients of the Taylor expansion of the functions $f(t)$, $\Phi(t)$ at the point t_0. The third relation (10) must be satisfied for arbitrary, including real, values of t_0. Since $\operatorname{Im} a_0 \neq 0$, it splits into two conditions, equivalent to (6),

$$ff'' - f'^2 = 0, \qquad (\varphi + \theta')' = 0,$$

under which Eqs. $(1'')$, $(2'')$ pass the Painlevé test.

Differentiating Eq. (1) and eliminating $\bar{u}(t)$, we can obtain the following equation of second order for $u(t)$:

$$u'' = \frac{u'^2}{u} - \lambda\overline{F}u^2 - i\varphi' u + F\varphi - iF' + \frac{F^2}{u}.$$

In the integrable case (6), the substitution $x = \mathrm{e}^{C_0 t}$, $y = \mathrm{e}^{-i\theta(t)}\,u$ transforms this equation into Eq. (7).

Differentiating Eq. (2) and eliminating $\bar{u}(t)$, we obtain the equation

$$u'' = \frac{\lambda u u'^2 - F'u' + \varphi(F\varphi - iF')u}{\lambda u^2 - F} - \lambda\overline{F}u^3 + (F\overline{F} - i\varphi')u$$

for the function $u(t)$. Under condition (6), the substitution

$$x = \mathrm{e}^{2C_0 t}, \qquad y = 1 - \frac{C_1}{\lambda u^2}\,\mathrm{e}^{C_0 t + 2i\theta(t)}$$

transforms this equation to Eq. (8). $\square$

Remark. By the substitution $\tilde{u} = \mathrm{e}^{-i\theta(t)}\,u$, Eqs. (1)–(4) reduce to similar equations in which the parameter $\theta(t)$ is zero. In other words, up to a substitution of the dependent variable, Eqs. (1), (2) are integrable for $f(t) = C_1\,\mathrm{e}^{C_0 t}$, $\varphi(t) = C_2$. In this case, the parameter $\varphi(t)$ does not belong to the class of functions increasing at infinity which appear in (1), (2) as $\varphi(t)$ in solving the autoresonance problem [6], [9].

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