The autoresonance threshold into a system of weakly coupled oscillators

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System of coupled oscillators

$$
\begin{aligned}
x'' + \omega^2 x &= \varepsilon \alpha_1 x y + \varepsilon(\gamma \exp\{i\varphi\} + c.c.) \\
y'' + (2\omega)^2 y &= \varepsilon \alpha_2 x^2,
\end{aligned}
$$

where $\varphi = (\omega + \varepsilon\alpha\tau)\theta,\ \tau = \varepsilon\theta$, ε is a small positive parameter. $\omega = const$ is a frequency of oscillator with an amplitude x, α_1 and α_2 are parameters of nonlinear coupling, γ is an amplitude of external perturbation, α is a derivative of detuning of frequency of external perturbation with respect to slow time τ.

Questions and standard answers

Question: Is it possible to change solutions over order $O(1)$ due to a small oscillating perturbation?

Standard answer:

A perturbation should be resonant for the linear system.

In nonlinear systems, the frequency of perturbation should vary slowly in such a way that there occurs an autoresonance in the system.

Autoresonance

Autoresonance phenomenon was found by Veksler and McMillan at 1944.

- **V.V. Veksler** New method for acceleration of relativist particles. Doklady AN SSSR. 1944. .43, 8. pp. 346-348.
- **E.M. McMillan** The synchrotron–a proposed high energy particle accelerator. Phys. Rev. 1945. 68, p.143

They show that to change amplitude of nonlinear oscillator one should slowly decrease frequency of external driver starting with resonant frequency.

Threshold of autoresonance

The autoresonance has many application. But it was found that the standard rule does not work for some systems with one and a half degree of freedom. In some cases there exists a threshold of amplitude for external driving force to the autoresonance occurs. Firstly this was observed by L.Friedland using numerical simulations for subharmonic modes at 2000 and by L. Kalyakin using asymptotical analysis for nonlinear equations with frequency for external driving force which quadratic decreases at 2001.

Up to now the results about threshold for autoresonance was known for some special systems with one and a half degree of freedom only.

Asymptotic reduction

Let us define new slow time $t = \varepsilon\sqrt{\alpha}\theta$. Solution of weakly coupled oscillating system will be construct in following form

$$x = \omega\sqrt{\frac{\alpha}{\alpha_1\alpha_2}}A(t)\exp(i(\omega\theta + \alpha\tau^2)) + c.c.,$$

$$y = \omega\frac{\sqrt{\alpha}}{\alpha_1}B(t)\exp(i(2\omega\theta + \alpha\tau^2)) + c.c..$$

where $A(t)$ and $B(t)$ are amplitudes of oscillating for (x, y); $f = \frac{\sqrt{\alpha_1\alpha_2}}{\alpha\omega}\gamma$.

Primary resonance system

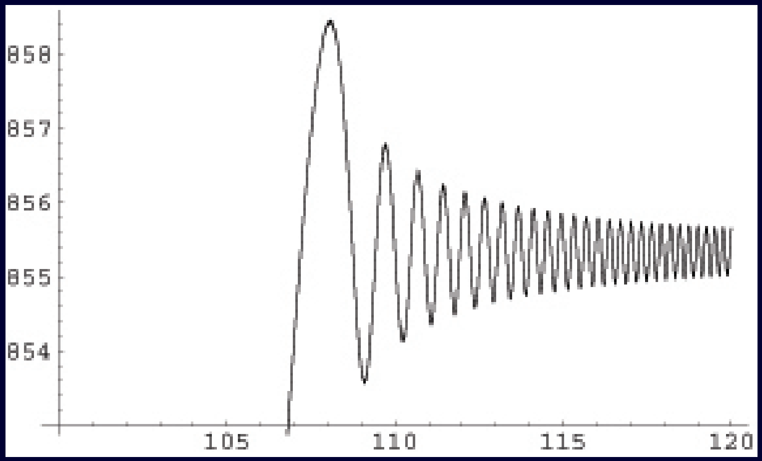
An asymptotic reductions give us following primary resonance system:

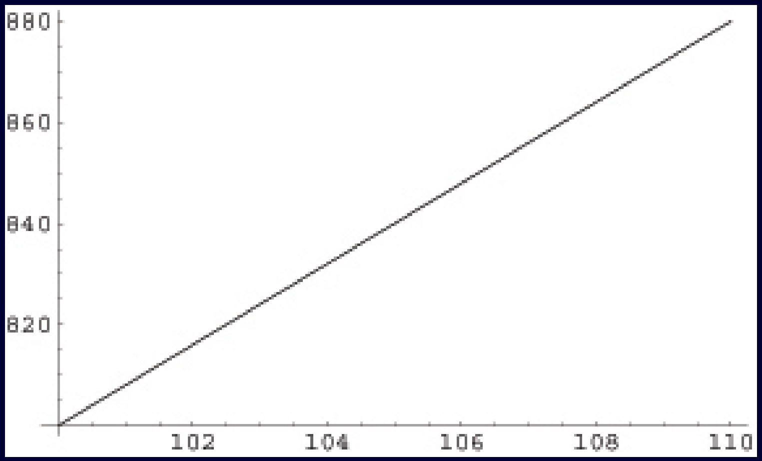
$$A'(t) = -i\left(2tA + \frac{1}{2}A^*B + f\right),$$

$$B'(t) = -i\left(4tB + \frac{1}{4}A^2\right).$$

Our aim is to study the behavior of the solutions for large values of t.

Numerical simulations





$f = 11.9$ $f = 12.1$

$$A(100) = 102.669 - i793.88, \quad B(100) = 386.825 + i101.831.$$

Here we show typical threshold phenomenon. For $|f| < 12$ numeric solution is bounded and for $|f| > 12$ there exists growing solution.

Algebraic solutions

Theorem. *When* $t \to \infty$, *there exists the solution of the system with the asymptotic behaviour of the form*

$$A_2(t) = -\frac{f}{2}t^{-1} + \frac{if}{4}t^{-3} + \Big(\frac{3f}{8} - \frac{f^3}{512}\Big)t^{-5} + O(t^{-7}),$$

$$B_2(t) = -\frac{f^2}{64}t^{-3} + \frac{7if^2}{256}t^{-5} + O(t^{-7}).$$

When $|f| \geq 12$,

$$A_1(t) = -8\Big(\cos(\Psi) + i\sin(\Psi)\Big)t + \frac{f}{4}t^{-1} + O(t^{-3}),$$

$$B_1(t) = -4\Big(\cos(2\Psi) + i\sin(2\Psi)\Big)t + (-\frac{f}{4} - 2i)t^{-1} + O(t^{-3}),$$

$$here\ \sin(\Psi) = \frac{12}{f}.$$

$$\begin{aligned} A_3(t) &= 8\Big(\cos(\Psi) + i\sin(\Psi)\Big)t + \frac{f}{4}t^{-1} + O(t^{-3}), \\ B_3(t) &= -4\Big(\cos(2\Psi) + i\sin(2\Psi)\Big)t \end{aligned}$$

$$+\Big(-\cos(\Psi)[\frac{f}{4} + \frac{24}{f}] + 2i[1 + \sin^2(\Psi)]\Big)t^{-1} + O(t^{-3}),$$

$$here\ \sin(\Psi) = -\frac{12}{f}.$$

A neighborhood of bounded solution $(A_2(t), B_2(t))$

$$A(t) = a(t)\exp\{-it^2\}, \quad B(t) = b(t)\exp\{-2it^2\}$$

$$\begin{aligned} ia'(t) &= \frac{1}{2}a^*b + f\exp\{it^2\}, \\ ib'(t) &= \frac{1}{4}a^2. \end{aligned}$$

We investigate solutions of the system in a neighborhood of $(A_2(t), B_2(t))$. Substitute

$$a = A_2\exp\{it^2\} + \alpha, \quad b = B_2\exp\{2it^2\} + \beta,$$

A neighborhood of bounded solution $(A_2(t), B_2(t))$

$$
\begin{aligned}
i\alpha' &= \frac{1}{2}\alpha^*\beta + \frac{1}{2}(A_2^*\beta\exp\{-it^2\} + \alpha^* B_2 \exp\{2it^2\}),\\
i\beta' &= \frac{1}{4}\alpha^2 + \frac{1}{2}A_2\alpha\exp\{it^2\}.
\end{aligned}
$$

Theorem. *There exists the formal asymptotic solution for the system. It has the form* $\alpha = \sum_{k=0}^{\infty}\alpha_k t^{-k}, \quad \beta = \sum_{k=0}^{\infty}\beta_k t^{-k}$. *The solution depends on four real parameters. The leading-order terms* α_0, β_0 *are determined in terms of elliptic functions.*

A neighborhood of increasing solution

$$(A_3(t), B_3(t))$$

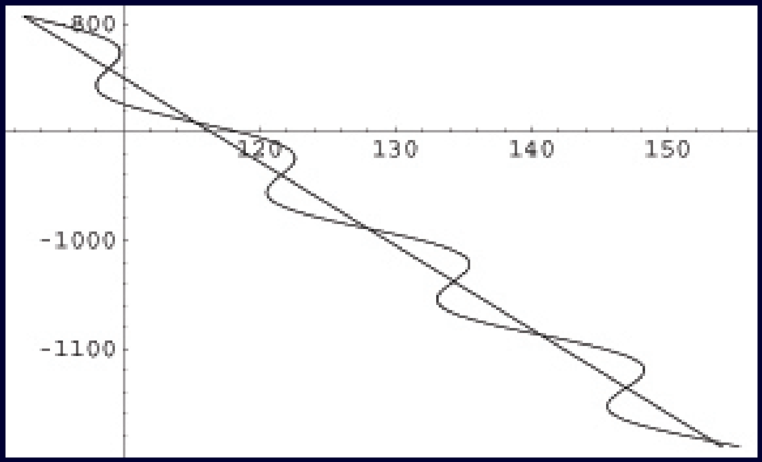
$$\begin{aligned} A'(t) &= -i\left(2tA + \frac{1}{2}A^*B + f\right), \\ B'(t) &= -i\left(4tB + \frac{1}{4}A^2\right). \end{aligned}$$

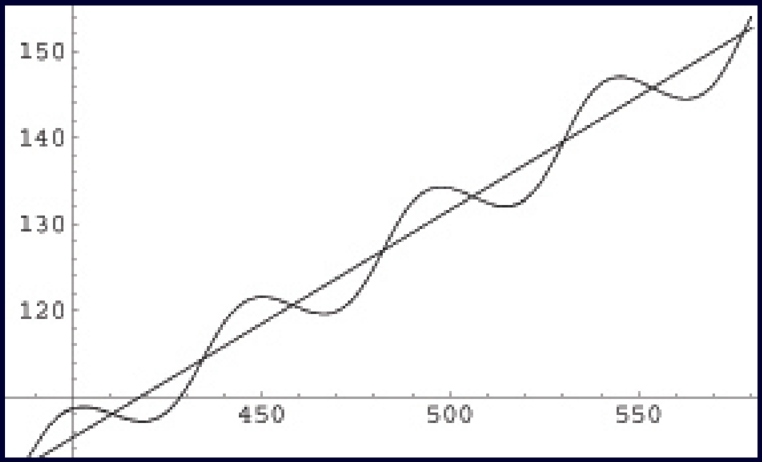
$$A|_{t=100} = A_3(100;2) + 0.1, \quad B|_{t=100} = B_3(100;2) + 0.1$$

amplitude of the perturbation is $f = 12.1$

Numerical simulations

Graphs are constructed on the complex plane A and B





$A_3(t)$ is the straight line and numerical solution $A(t)$

$B_3(t)$ is the straight line and numerical solution $B(t)$

Relative value of difference between the numerical solution and algebraic solution

relative value of difference: $\left| \frac{A(t) - A_3(t)}{A(t)} \right|$

Results

- It is shown that there exist increasing and bounded solutions when $|t| \to \infty$.
- Bounded solutions are studied in detail.
- It is shown that the periodic perturbation can lead to the capture into resonance.
- We present the asymptotic description of the capture.
- We obtain explicit formulas for the threshold values of the perturbation amplitude.

References

When we have put our preprint at http://arXiv.org:

- S.Glebov, O.Kiselev, V.Lazarev. The autoresonance threshold into a system of weakly coupled oscillators. arXiv:07072311[math-ph] 16 Jul 2007.

and send the preprint to our colleagues we have receive an answer from L Friedland. He send us his new work published in the same time (August 7, 2007):

- O.Yaakobi, L. Friedland and Z. Heinis. Driven, autoresonant three-oscillator interactions. Physical Review E, v.76, 026205 (2007)

where was found the threshold for three coupled oscillators.

THANK You