The capture into parametric autoresonance

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Outlines

- parametric autoresonance in nonlinear equations;
- capture into parametric resonance;
- scattering problem for captured solutions;
- perturbation theory and matching method;

Parametric resonance

Our subject is a nonlinear equations, for example parametric driven pendulum equation:

$$\phi'' + (1 + \varepsilon \cos(\omega t + \phi_0)) \sin(\phi) = 0, \quad \omega = \text{const}, \quad 0 < \varepsilon \ll 1. \quad (1)$$

Solutions with small amplitude have behaviors close to solution of parametrically driven linear equation. There exist the set of the resonant gap on the plane of parameters (ω, ε). The solutions grow exponentially for this parameters [see Floquet, 1883]. The most wide resonant gap corresponds to $\omega = 2 + \sigma\varepsilon$.

Equations of primary parametric resonance

Nonlinearity leads to the unbalancing of the frequency of external force and frequency of oscillations for the solution of the nonlinear equation. Perturbation theory says that small oscillating solutions has order $\sqrt{\varepsilon}$. Amplitude R and phase shift ν of such solutions are defined by primary parametric resonance equation [see Bogolyubov and Mitropolskii]:

$$\frac{dR}{d\tau} + R\sin(2\nu) = 0, \quad \frac{d\nu}{d\tau} - \sigma - R^2 + \cos(2\nu) = 0, \quad \tau = t\varepsilon \quad (2)$$

This system is equivalent to equation:

$$i\frac{d\phi}{d\tau} + (|\phi|^2 + \sigma)\phi - \phi^* = 0, \quad \phi = R\exp\{i\nu\}.$$

Autoresonance problem

Equation (2) have solutions this means that the resonant grows is suppressed by nonlinearity. The problem is:

- How to increase the amplitude oscillations in nonlinear systems by small driving force?

The answer was done by physicist Veksler and McMillan independently at 1944.

- Decrease frequency of perturbation slowly!

Review

The capture in a nonlinear (not parametric) resonance was studied by Chirikov, 1959. The capture in the nonlinear resonance is associated with loss of stability and slow crossing through the separatrix, see Neishtadt, 1975. The autoresonance was observed in many different oscillatory and wave processes (see review by Friedland, 2005). Asymptotic approach to the autoresonant solution was considered in Kalyakin, 2003. Full asymptotic expansions for the autoresonant grow of solution up to order 1 were done by Garifullin, 2004.

Simplest case

The way given by Veksler and McMillan leads to study the equations for primary parametric resonance with varying σ. Such equations were studied by physicist Fajans, Gilson, Friedland, Khain, Meerson, Asaf at 2000-2005, when $\sigma \not\equiv$ const. It was shown that the autoresonance phenomenon takes place for the primary parametric resonance equation when $\sigma < 0$ and $|\sigma|$ grows with respect to τ.

We consider equation for primary parametric autoresonance in simplest form:

$$i\varepsilon\frac{d\phi}{d\theta} + (-\theta + |\phi|^2)\phi - \phi^* = 0, \quad \theta = \varepsilon\tau \tag{3}$$

Reasons

There are two reasons to investigate equation (3).

- This case $\sigma(\theta) \equiv -\theta$ is the simplest one and it contains the autoresonance phenomenon for parametrically driven systems.
- This case saves most of essential features of the solution with general dependency of $\sigma(\theta)$.

Problems

Let us consider two numeric solutions for equation (3) (see Figure 1). These solutions differ at the initial moment only. The first solution (left) corresponds to the initial moment at $\theta = -2$ and the second one (right) corresponds to $\theta = -2.01$. We see the solution of the equation is very sensitive with respect to initial data.

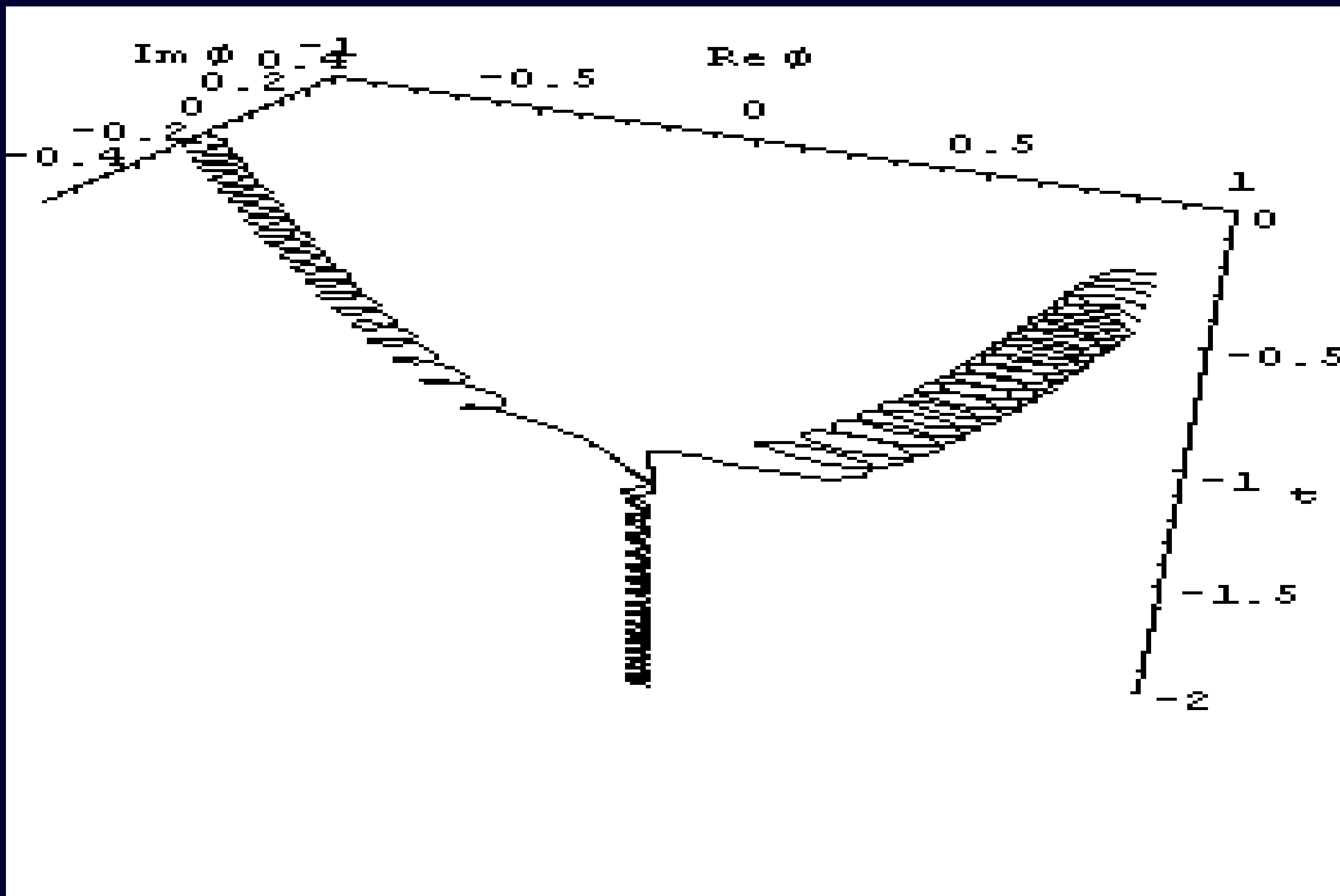


Figure 1: The figure shows two solution of (3). The left curve shows the solution of the Cauchy problem for equation (3) at $\theta = -2.01$, $\Re(\phi) = 0.02$, $\Im(\phi) = 0$, $\varepsilon = 0.01$. The right curve shows the solution of the Cauchy problem for equation (3) at $\theta = -2$, $\Re(\phi) = 0.02$, $\Im(\phi) = 0$, $\varepsilon = 0.01$.

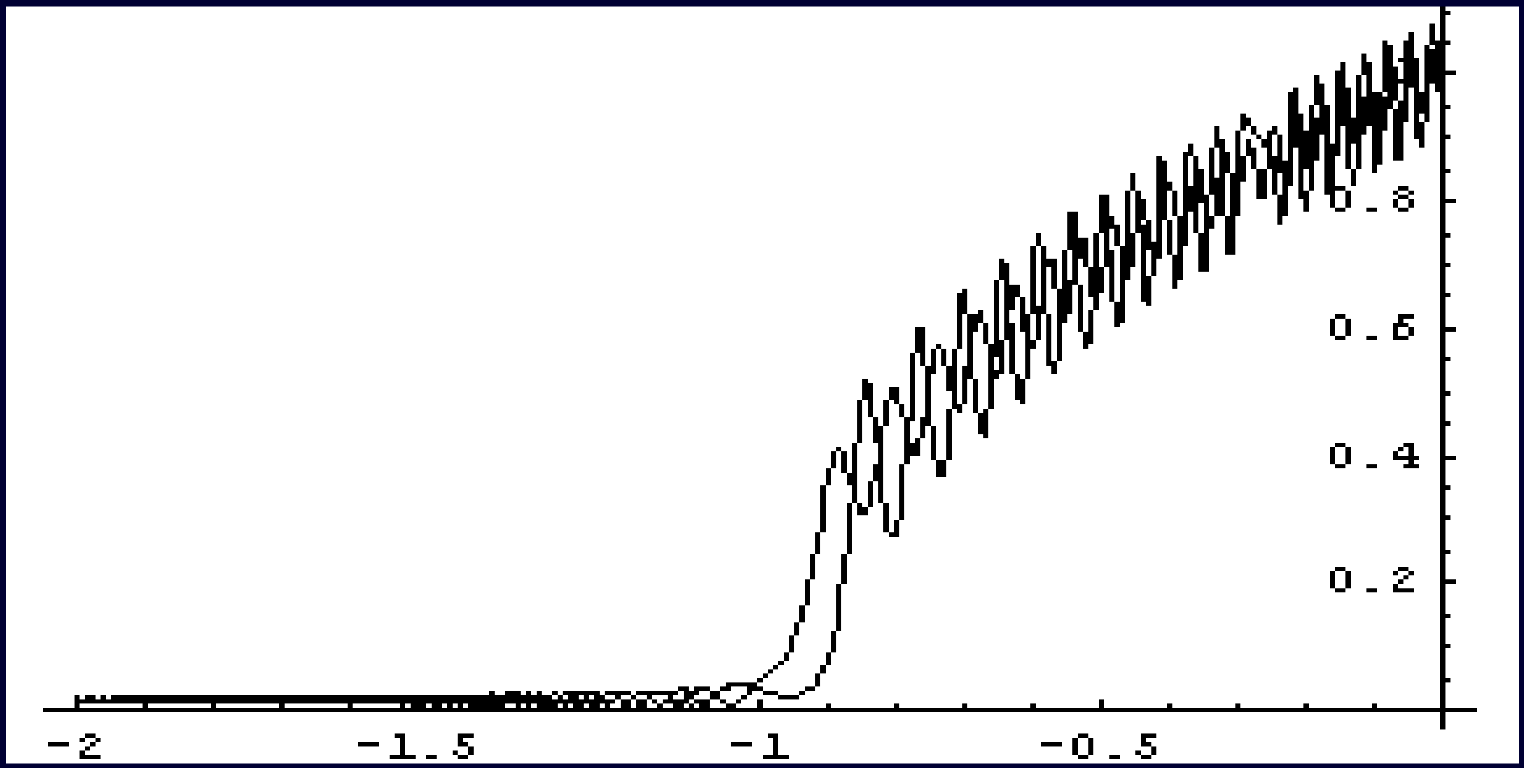


Figure 2: This figure shows $|\phi|^2$ for the solutions of the Cauchy problem for equation (3) at $\theta = -2$, $\Re(\phi) = 0.02$, $\Im(\phi) = 0$, $\varepsilon = 0.01$ and at $\theta = -2.01$, $\Re(\phi) = 0.02$, $\Im(\phi) = 0$, $\varepsilon = 0.01$.

Goals

Our goals:

- to develop an asymptotic theory of the capture into the parametric autoresonance.

It means

- to give asymptotic formulas for solutions before and after the capture;
- to find the connection formulas for the asymptotics before and after the capture.

Approach

The capture into parametric autoresonance may be regarded as loss of stability in the pitchfork bifurcation. The dynamic theory of the pitchfork bifurcation for second order ordinary differential equations with slowly varying coefficients was considered by Haberman,1979,2001; Maree,1996. The solution in bifurcation layer is approximated by solutions of the Painlevé-2 equation Haberman. Later on the connection formulas were obtained for the solution before and after supercritical pitchfork bifurcation for an asymptotic solution of the special form of perturbed Painlevé-2 equation with dissipation and slowly varying bifurcation parameter Maree, 1996. These formulas were based on connection formulas for solutions of Painleve-2 equations obtained by Its and Kapaev,1989.

Pitchfork bifurcation

Here we present the qualitative analysis for the equation with the "frozen" coefficient $\sigma \equiv -T$:

$$i\epsilon\phi' + (-T + |\phi|^2)\phi - \phi^* = 0. \qquad (4)$$

The trajectory of the solution for the equation with varying coefficient $\sigma(\theta) = -\theta$ is locally close to the trajectories of equation (4) at $T = -\theta$. Therefore the solution of the equation with the frozen coefficient gives a local behavior for the solution of (3).

The Hamiltonian for the equation is:

$$H(\phi, \phi^*) = -\frac{1}{2}|\phi|^4 + T|\phi|^2 + \frac{1}{2}((\phi^*)^2 + \phi^2).$$

It is easy to see that there exists only one center when $T < -1$. There are two centers and one saddle point when $-1 < T < 1$ and three centers and two saddle points for $T > 1$. We show the phase portrait for the equation with different values of the parameter T on the following figure.

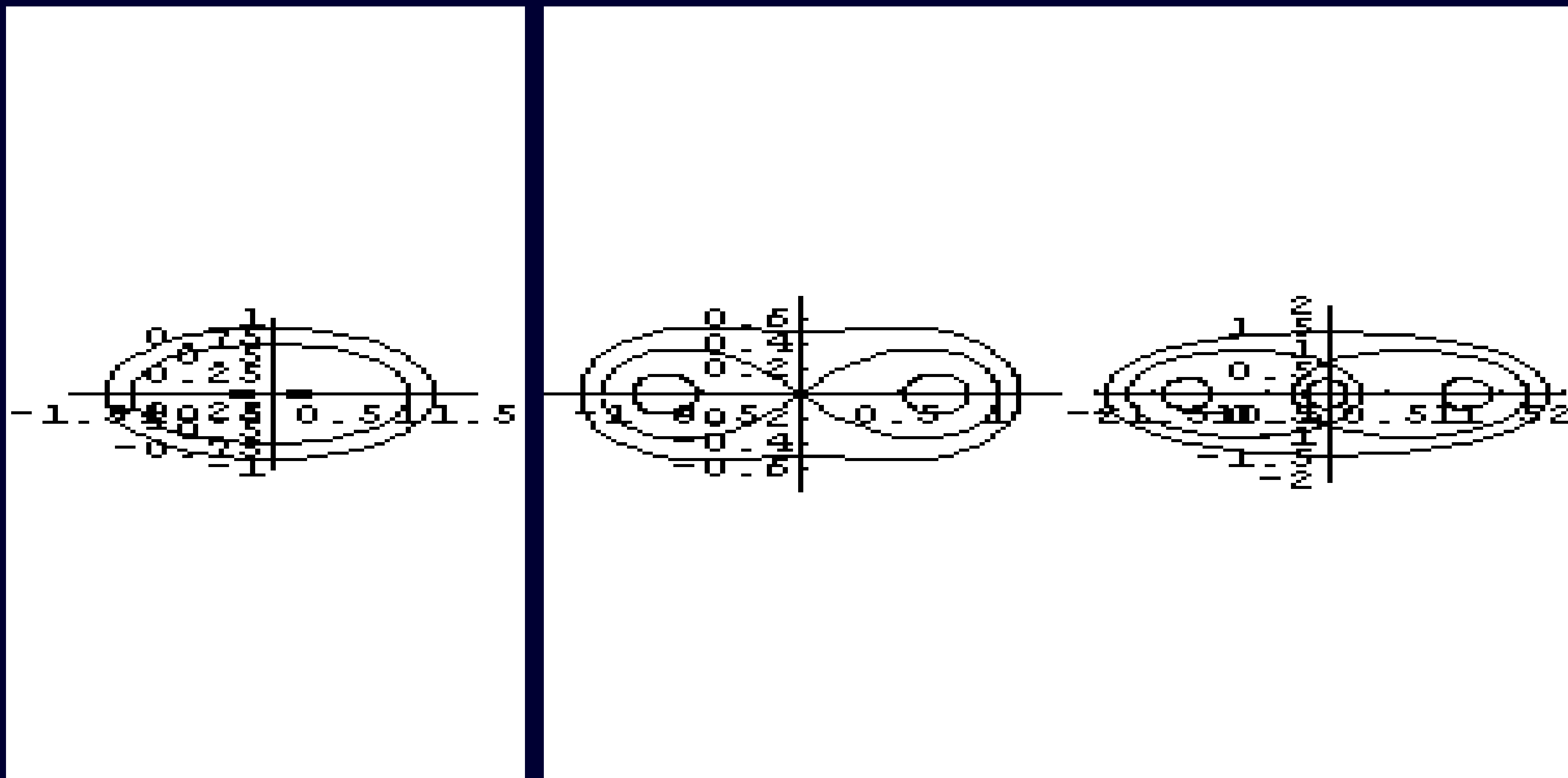


Figure 3: On the left picture $T < -1$, on the middle picture $T = 0$, and on the right picture $T > 1$.

Asymptotic solution before capture

Let us construct a solution of the WKB-type before the capture. A qualitative analysis shows that the capture into the parametric resonance occurs at $\theta = -1$. When $\theta < -1$ the solution of equation (4) has a unique center at $\phi = 0$. This fact prompts that equation (3) has oscillating solutions and they are close to $\phi = 0$.

WKB expansion

Let us consider a WKB-solution of (3) in the form

$$\phi = \varepsilon^{1/2} \sum_{n=1}^{\infty} \varepsilon^{n-1} \phi_n(s, \theta). \qquad (5)$$

The leading term of the asymptotic expansion is a solution of the equation

$$i\omega' \partial_s \phi_1 - \theta \phi_1 - \phi_1^* = 0. \qquad (6)$$

The higher-order terms of (5) are solutions of

$$i\omega\phi_n' - \theta\phi_n - \phi_n^* = f_n, \quad n = 2, 3, \ldots, \tag{7}$$

where

$$f_n = -i\partial_\theta\phi_{n-1} - i\partial_\theta\varphi_{n-1}\partial_s\phi_n + h_n.$$

Here h_n is a polynomial of the third order with respect to ϕ_m, $m < n$.

Asymptotic behavior close to the turning point

The WKB asymptotic expansion is not valid at the turn point $\theta = -1$. As $\theta \to -1-0$ the asymptotic behavior of the coefficients for asymptotic series (5) is described by the following formulas. For the leading term we obtain

$$\phi_1 \sim \alpha_{1,0}\left(\frac{-2}{\theta+1}\right)^{1/4} \sin\left(-\frac{2\sqrt{2}}{3}(-1-\theta)^{3/2}-(\alpha_{1,0})^2\ln(1+\theta)+\varphi_{1,0}\right),$$

as $\theta \to -1-0$, where $\alpha_{1,0} = \text{const}$ and $\varphi_{1,0} = \text{const}$ are parameters of the solution.

The second term of asymptotic series (5) behaves like

$$\phi_2 = O((-1-\theta)^{-7/4}) \quad \text{as } \theta \to -1-0.$$

Recurrent calculations give

$$\phi_{n+1} = O(\phi_n(-1-\theta)^{-3/2})), \quad \text{as } \theta \to -1-0.$$

The domain of validity

Expansion (5) preserves the asymptotic property with respect to ε when

$$\frac{\varepsilon\phi_{n+1}}{\phi_n} \ll 1, \quad \text{as} \quad \theta \to -1-0.$$

It yields the domain of validity for (5), namely

$$\varepsilon^{-2/3}(-1-\theta) \gg 1.$$

The Painlevé layer

To construct an asymptotic solution when θ is close to -1 we use the scaled variables:

$$\theta + 1 = \varepsilon^{2/3}\eta, \quad \phi = \varepsilon^{1/3}x(\eta,\varepsilon) + i\varepsilon^{2/3}y(\eta,\varepsilon).$$

This change of variables leads to the system of equations for the real and imaginary parts of the function ϕ:

$$x' + 2y = \varepsilon^{2/3}(\eta - x^2)y - \varepsilon^{4/3}y^3, \quad y' + (\eta - x^2)x = \varepsilon^{2/3}y^2x. \quad (8)$$

The asymptotic expansion in the Painlevé layer

We construct a solution of system (8) of the form

$$x(\eta,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^{2n/3} x_n(\eta), \quad y(\eta,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^{2n/3} y_n(\eta). \qquad (9)$$

The leading term of the asymptotic expansion x_0 is a solution of the second order equation:

$$x_0'' + 2(-\eta + x_0^2)x_0 = 0. \qquad (10)$$

The higher-order terms of (9) satisfy to the second order linear differential equation

$$x_n'' + 2(-\eta + 3x_0^2)x_n = h_n. \tag{11}$$

The substitutions

$$\eta = 2^{1/3}z, \quad x_0(\eta) = 2^{1/3}iu(z) \tag{12}$$

reduce (10) to the Painleve-2 equation:

$$u'' - zu - 2u^3 = 0. \tag{13}$$

The Painlevé-2 equation is integrable by the isomonodromic deformation method See Flaschka, Newell, 1980. The solution $u(z;\tilde{\alpha},\tilde{\varphi})$ of the equation is called Painlevé transcendent, depending on two parameters $\tilde{\alpha}$ and $\tilde{\varphi}$. It is known that the real solution of (13) has no singularities. Therefore, the main term of the asymptotic expansion is presented by the Painlevé transcendent $u(z,\tilde{\alpha},\tilde{\varphi})$.

The asymptotic behavior at the left border of the validity interval

The asymptotic expansion of the Painlevé-2 solution as $z \to -\infty$ has a form (see Its, Kapaev,1989 and Belogrudov,1997)

$$u(z) = i\tilde{\alpha}(-z)^{-1/4}\sin(\frac{2}{3}(-z)^{3/2} + \frac{3}{4}\tilde{\alpha}^2\ln(-z) + \tilde{\varphi}) + o((-z)^{-1/4}). \quad (14)$$

Here $\tilde{\alpha}$ and $\tilde{\varphi}$ are parameters of the solution of Painlevé-2 equation.

The left border of the validity interval for the Painlevé layer

Using the asymptotic behavior of x_0, as $\eta \to -\infty$ we obtain

$$x_n^- = O((-\eta)^{n-1/4}).$$

This value for the higher-order terms gives the left border for the interval of validity of (9):

$$\frac{\varepsilon^{2(n+1)/3} x_{n+1}}{\varepsilon^{2n/3} x_n} \ll 1, \quad \varepsilon^{2/3}(-\eta) \ll 1,$$

Matching with the WKB asymptotic expansion

The domains of validity of asymptotic expansions (5) and (9) meet each other. The matching with the WKB asymptotic expansion allows one to obtain the parameters $\tilde{\alpha}$ and $\tilde{\varphi}$ for the solution of the Painlevé-2 equation, for example,

$$\tilde{\alpha} = \alpha_{1,0}, \quad \tilde{\varphi} = \varphi_{1,0}. \tag{15}$$

Asymptotic behavior at the right border

The theorem by Its-Kapaev-Belogrudov gives the asymptotic behavior of solution (14) for the Painlevé-2 equation.

If a solution of the Painlevé 2 equation has asymptotic behavior (14), as $z \to -\infty$, and

$$\tilde{\varphi} = \frac{3}{2}\tilde{\alpha}^2 \ln(2) - \frac{\pi}{4} - \arg\left(\Gamma\left(\frac{i\tilde{\alpha}^2}{2}\right)\right) + \kappa\pi(\mod (2\pi)), \quad \kappa = 0, 1,$$

then the solution of the Painlevé-2 equation is:

$$u(z) = \frac{iq}{2\sqrt{\pi}} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2} \right) (1 + o(1)), \quad z \to \infty,$$

where $q^2 = \exp(\pi\tilde{\alpha}^2) - 1$, and $\mathrm{sgn}(q) = 1 - 2\kappa$;
Otherwise the absolute value of the solution increases:

$$u(z) = \pm i\sqrt{\frac{z}{2}} \pm i(2z)^{\frac{-1}{4}} \rho \cos\left(\frac{2\sqrt{2}}{3} z^{3/2} - \frac{3}{2}\rho^2 \ln(z) + \upsilon \right) + o(z^{-1/4}), \tag{16}$$

where $\rho > 0$ and $0 \leq \upsilon < 2\pi$. The parameters ρ and υ are determined by $\tilde{\alpha}$ and $\tilde{\varphi}$,

$$\rho^2 = \frac{1}{\pi} \ln\left(\frac{1+|p|^2}{2|\Im(p)|}\right),$$

$$v = -\frac{3\pi}{4} - \frac{7}{2}\rho^2 \ln(2) + \arg(\Gamma(i\rho^2) + \arg(1+p^2),$$

where

$$p = (\exp(\pi\tilde{\alpha}^2 - 1)^{1/2} \exp\left(i\frac{3}{2}\tilde{\alpha}^2 \ln(2) - i\frac{\pi}{4} - i\arg(\Gamma(i\frac{\tilde{\alpha}^2}{2})) - i\tilde{\varphi}\right),$$

and the sign "$+$" is defined by $\Im(p) < 0$;

if $\rho = 0$, then

$$u(z) = \pm i\sqrt{\frac{z}{2}} \pm i\frac{z^{-5/2}}{8\sqrt{2}} + O(z^{-11/2}). \tag{17}$$

These formulas give an asymptotic solution of system (8) as $\eta \to \infty$. The solution of the scattering problem for system (8) allows one to solve the problem of capture into the resonance for the small amplitude solution of the primary resonance equation.

The right border of validity interval for the asymptotic expansions in Painlevé layer

Formulas (16), (17) and substitutions (12) give the leading term of the asymptotic solution captured into the resonance. The solution has the behavior

$$x_0(\eta) \sim \mp 2^{1/3}\sqrt{\eta}, \quad \eta \to \infty.$$

The nonlinear terms in the perturbed Painlevé-2 equation (8) are

$$x_n(\eta) = O((-\eta)^{n+1/2}).$$

Then asymptotic expansion (9) is valid in the domain

$$\varepsilon^{2/3}\eta \ll 1.$$

The captured WKB-asymptotic solution

The previous section shows that the asymptotic solution of equation (3) increases like $\pm\sqrt{\theta}$. Equation (3) has two slowly varying solutions for $-1 < \theta < 1$. In this section we construct the formal asymptotic expansion for these solutions and for slowly varying solutions if $\theta > 1$, too.

Slowly varying solutions

Let us to construct a solution of equation (3) as follows:

$$U(\theta, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n U_n(\theta). \tag{18}$$

The main term of the asymptotic expansion is defined by the algebraic equation

$$-\theta U_0 + |U_0|^2 U_0 - U_0^* = 0. \tag{19}$$

Using the complex conjugated function we obtain:

$$[U_0 + U_0{}^*][U_0 - U_0{}^*] = 0.$$

This formula shows that the leading term of the asymptotic expansion is purely real or purely imaginary. The real terms are

$$U_0^{(1)} = 0, \quad U_0^{(2,3)} = \pm\sqrt{1+\theta}, \tag{20}$$

and the imaginary terms are

$$U_0^{(4,5)} = \pm i\sqrt{\theta - 1}. \tag{21}$$

When $\theta \to \infty$ the order of the n-th term of the asymptotics is $O(\theta^{-1/2})$, as $n = 1, 2, \ldots$. Therefore the asymptotic expansion (18) is uniform with respect to θ when $\theta > -1$, for $j = 2, 3$, and when $\theta > 1$, for $j = 4, 5$.

WKB-asymptotic expansion close to the slowly varying centers

Here we construct the WKB-asymptotic expansion that oscillates close to the slowly varying centers $U^{(j)}$. The WKB asymptotic expansion has the form

$$\Phi(\theta,\varepsilon) = U^{(j)}(\theta,\varepsilon) + \sum_{k=1}^{\infty} \varepsilon^{k/2} \Phi_k^{(j)}(S,\theta,\varepsilon). \qquad (22)$$

Here

$$S = \Omega(\theta)/\varepsilon + \sum_{k=0}^{\infty} \varepsilon^{k/2} \varphi_k(\theta).$$

Equations for the higher-order terms are

$$i\Omega'\partial_S\Phi_k^{(j)} + [-\theta + 2|U_0^{(j)}|^2]\Phi_k^{(j)} + [(U_0^{(j)})^2 - 1]\Phi_k^{(j)*} = G_n, \quad (23)$$

$$G_n = -i\sum_{l+k=n-2}\partial_S\Phi_l^{(j)}\varphi_l' - i\partial_\theta\Phi_{n-2}^{(j)} - \sum_{k+l+m=n} C_{klm}(U_0^{(j)})\Phi_k^{(j)}\Phi_l^{(j)}\Phi_m^{(j)*},$$

where $C_{klm}(U_0^{(j)})$ are polynomials. As a result we derive an asymptotic formula for the solution, as $\varepsilon \to 0$:

$$\Phi^{(j)}(S,\theta,\varepsilon) \sim (-1)^j \Bigg[\frac{1}{2}\sqrt{1+\theta} + \varepsilon^{1/2} A_{0,0} \left(\frac{1}{\sqrt[4]{1+\theta}} \cos(S) + i \sqrt[4]{1+\theta} \sin(S) \right) \Bigg],$$

$$S \sim \frac{\Omega}{\varepsilon} + \varphi_{0,0} - 2A_{0,0}^2 \Big(\theta + 3\ln(1+\theta) \Big).$$

Here $A_{0,0}$ and $\varphi_{0,0}$ are parameters of the constructed WKB-solution.

The domain of validity for the WKB-asymptotic solution at short distance from the slowly varying equilibrium.

The constructed WKB-asymptotic solution is not valid at the turning points. The higher-order terms of the asymptotic expansion are infinite at these points. The turning points are $\theta = -1$ and $\theta = \infty$. Using explicit formulas for the higher-order terms we deduce that

$$\Phi_n^{(j)} = O((\theta+1)^{3/2}\Phi_{n-1}^{(j)}), \quad \theta \to -1+0.$$

Then the domain of validity for the WKB-solution is

$$\varepsilon^{-2/3}(1+\theta) \gg 1, \quad \theta \to 1+0.$$

When $\theta \to \infty$ the domain of validity for the WKB-solution is obtained analogously,

$$\Phi_n^{(j)} = O(\theta^{5/4}\Phi_{n-1}^{(j)}), \quad \theta \to \infty,$$

whence

$$\theta \ll \varepsilon^{-4/5}, \quad \theta \to \infty.$$

Matching the expansion in the Painlevé layer and the captured WKB-asymptotic expansion

Asymptotic expansion (9) is valid for $-\varepsilon^{2/3}\eta \ll 1$ or $(\theta+1) \ll 1$ and θ close to -1. The captured asymptotic expansion is valid for $\varepsilon^{-2/3}(1+\theta) \gg 1$. Therefore, captured asymptotic solution (22) and asymptotic expansion (9) are both valid in the domain $\varepsilon^{2/3} \ll (1+\theta) \ll 1$. Using the uniqueness theorem for the asymptotic expansions we conclude that these expansions coincide in this domain. The usual matching procedure gives us connection formulas for the parameters A_k and ϕ_k of expansion (22) and parameters of expansion (9). For example, the parameters of the first correction term of (22) are

$$\varphi_{0,0} = v, \quad A_{0,0} = \rho. \tag{24}$$

Main result

Let us formulate the main result. Let the asymptotic solution of primary resonance equation (3) be

$$\phi \sim \varepsilon^{1/2}\alpha_{1,0}\left[\sqrt[4]{\frac{\theta-1}{\theta+1}}\sin(s) + i\sqrt[4]{\frac{\theta+1}{\theta-1}}\cos(s)\right], \quad \text{for } \theta < -1, \tag{25}$$

where

$$s = \frac{\omega}{\varepsilon} + \varphi_{1,0} + \alpha_{1,0}^2\left[\theta + \frac{3}{4}\ln\left|\frac{\theta-1}{\theta+1}\right|\right],$$

and

$$\omega = \frac{1}{2}\theta\sqrt{\theta^2-1} + \frac{1}{2}\ln(-\theta+\sqrt{\theta^2-1}).$$

The constants $\alpha_{1,0}$ and $\varphi_{1,0}$ are parameters of the solution and

$$\varphi_{1,0} \neq \frac{3}{2}\alpha_{1,0}^2 \ln(2) - \frac{\pi}{4} - \arg\left(\Gamma(i\alpha_{1,0}^2/2)\right) + \chi\pi(\mod (2\pi)), \chi = 0, 1.$$

Then, the asymptotic solution has the form:

$$\phi \sim \mu\left[\frac{1}{2}\sqrt{1+\theta} + \varepsilon^{1/2} A_{0,0}\left(\frac{1}{\sqrt[4]{1+\theta}}\cos(S) + i\sqrt[4]{1+\theta}\sin(S)\right)\right], \quad (26)$$

$$S \sim \frac{\Omega}{\varepsilon} + \varphi_{0,0} - 2A_{0,0}^2\left(\theta + 3\ln(1+\theta)\right),$$

$$\mu = -\mathrm{sign}(\Im(p)),$$

in the domain $\theta > -1$. Here

$$\Omega \sim \frac{4}{3}(\theta+1)^{3/2},$$

the constants $\varphi_{0,0}$ and $A_{0,0}$ are parameters of the asymptotic solution defined by

$$A_{0,0}^2 = \frac{1}{\pi}\ln\left(\frac{1+|p|^2}{2|\Im(p)|}\right),$$

$$\varphi_{0,0} = -\frac{3\pi}{4} - \frac{7}{2}A_{0,0}^2\ln(2) + \arg\left(\Gamma(iA_{0,0}^2)\right) + \arg(1+p^2),$$

where

$$p = \sqrt{\exp(\pi\alpha_{1,0}^2) - 1}\,\exp\left(i\frac{3}{2}\alpha_{1,0}^2\ln(2) - i\frac{\pi}{4} - i\arg\left(\Gamma(i\alpha_{1,0}^2/2)\right) - i\varphi_{1,0}\right).$$

Comparison between asymptotic and numerical solutions

Here we present series of figures with comparing of numerical and asymptotic solutions for the problem.

We solve the Cauchy problem for equation (3) at $\varepsilon = 0.0001$. The initial datum is related to the leading term of WKB-asymptotic solution at $\theta = -3$ and parameters $\alpha_{1,0} = 0.4$, $\varphi_{1,0} = 0$. It yields:

$$\phi^{num}|_{\theta=-3} = 0.00437617 + i0.00131843. \quad (27)$$

The real part of the numerical solution is presented on figure 4.

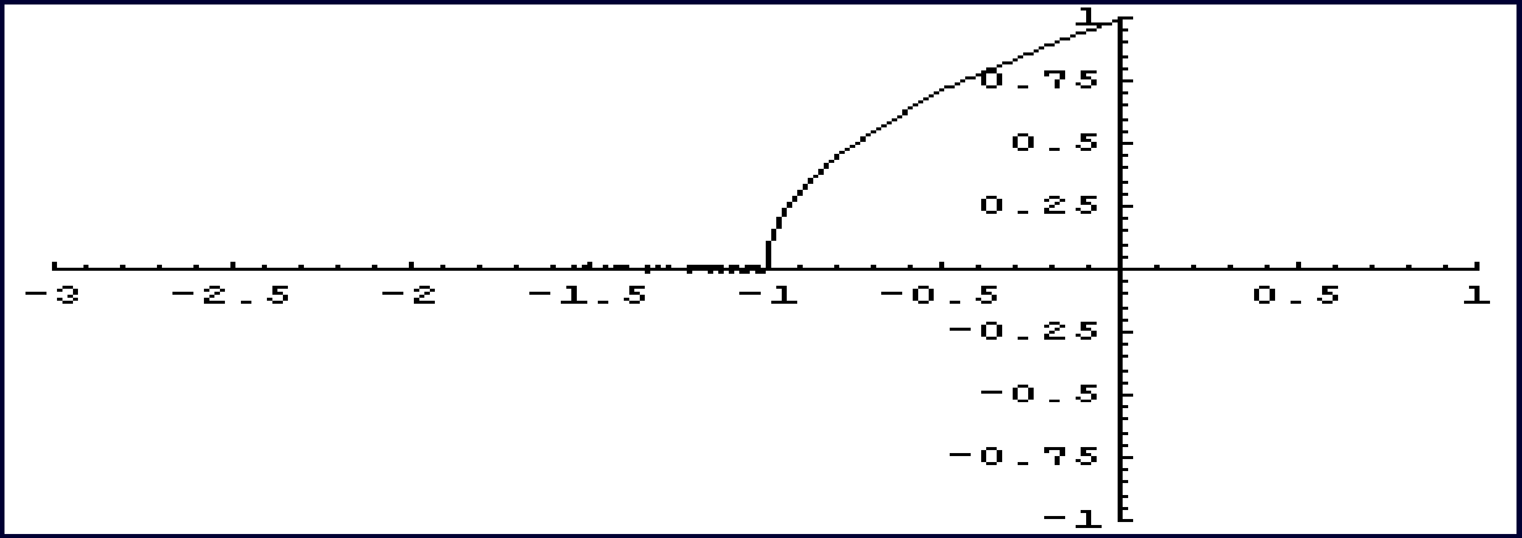


Figure 4: The real part of numerical solution for Cauchy problem (3),(27) at $\varepsilon = 0.0001$.

Our result predicts the domains of initial data as $\theta < -1$ which correspond to $\phi \sim \pm\sqrt{1+\theta}$ as $\theta > -1$. It is shown on figure (5) as $\theta = -3$.

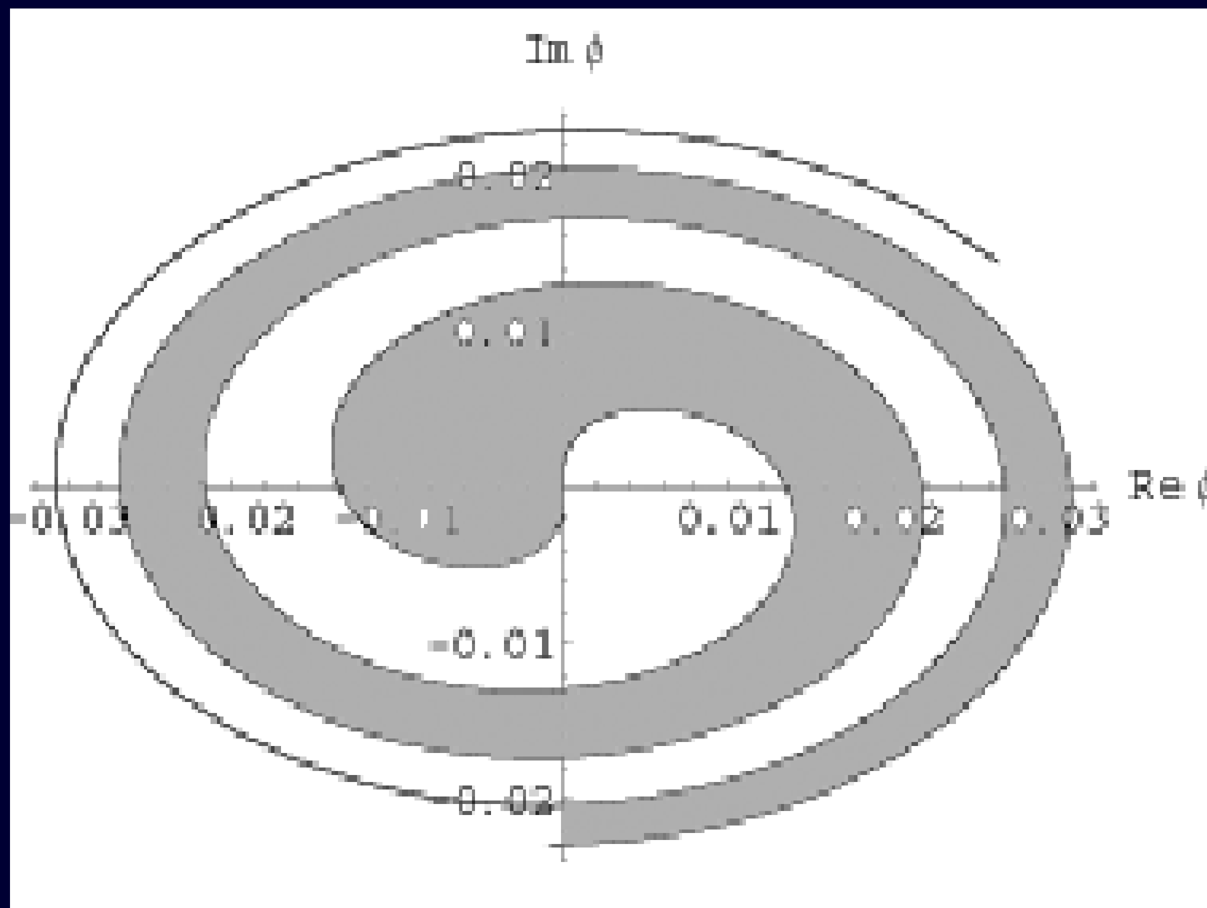


Figure 5: The domains of attraction of parametric resonance capture for solution of (3),(27) at $\varepsilon = 0.0001$ and $\theta = -3$. The white color corresponds to solutions that tend to $\sqrt{1+\theta}$ when $\theta > -1$ and the gray color corresponds to solutions that tend to $-\sqrt{1+\theta}$ when $\theta > -1$.

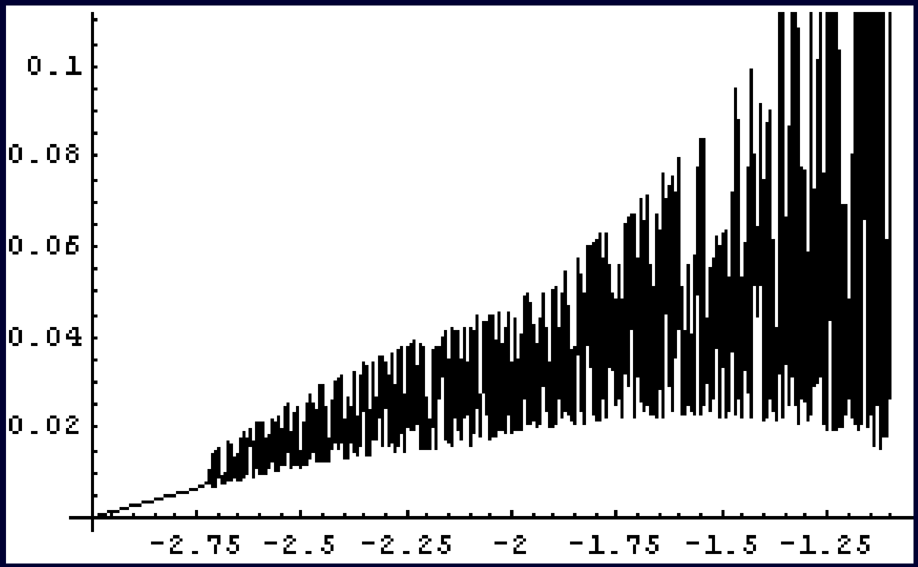
To compare results of Theorem 1 and numerical solution of (3),(27) we calculate the parameters of WKB-asymptotic solution (26) $\mu, A_{0,0}$ and $\varphi_{0,0}$ using given $\alpha_{1,0} = 0.4$ and $\varphi_{1,0} = 0$. It yields

$$\mu = 1, \quad A_{0,0} = 0.250149, \quad \varphi_{0,0} = -3.43149.$$

Then we calculate a comparative accuracy

$$\delta(\theta) = \frac{|\phi^{num} - \phi|}{|\phi^{num}|}, \quad \theta < -1,$$
$$\delta(\theta) = \frac{|\phi^{num} - \phi|}{|\phi^{num} - \sqrt{\theta + 1}|}, \quad \theta > -1.$$

Next figure gives the comparative accuracy in domains before and after the capture.



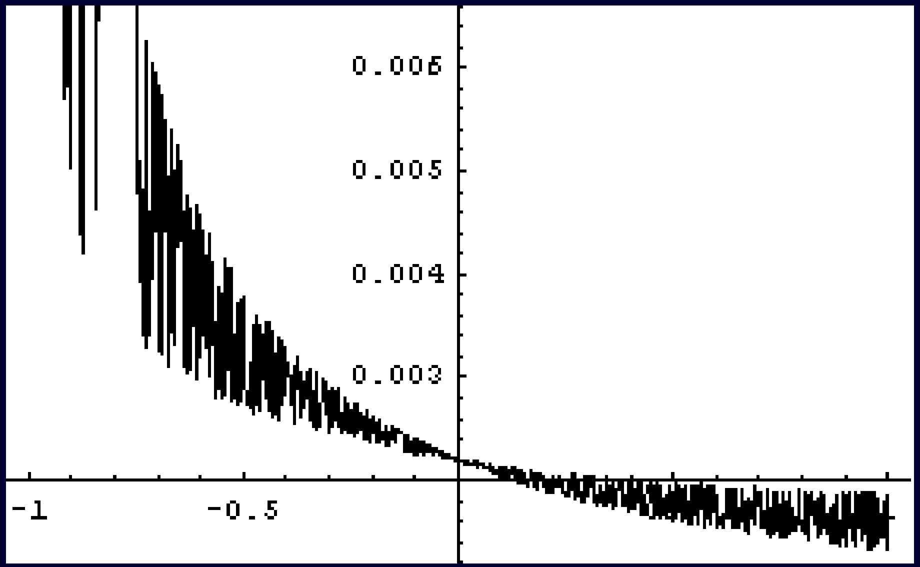


Figure 6: The comparative accuracy with WKB-asymptotic solution. On the left picture $\theta < -1$ and on the right picture $\theta > -1$.